Turing's 1935: my guess about his intellectual journey to "On Computable Numbers"

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Turing's 1936 Paper

"*On Computable Numbers, with an Application to the Entscheitungsproblem*" *Proceedings of the London Mathematical Society, ser.2, vol.42 (1936-37). pp.230-265; corrections, Ibid, vol 43(1937) pp.544-546*

- shows a variant of Gödel's Incompleteness proof(1931)
- contains the blueprint of computer (Universal Machine)

My Motivation

Curious: how did Turing get the ideas underlying this foundational paper of modern computer?

- a computer = Universal Machine
- a computer = a single machine that can do any mechanical computation

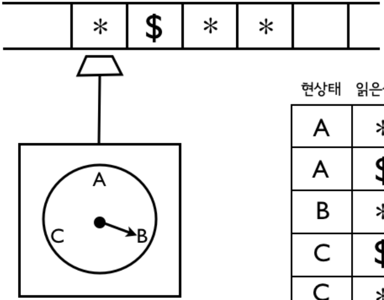
This talk:

- the content of the 1936's paper and
- its intellectual "pedigree" (my guess)

Turing's 1936 Paper

Theorem. By mechanical way we cannot generate all true formulas.

mechanical computation $\stackrel{\text{def}}{=}$ execution by a turing machien(TM)



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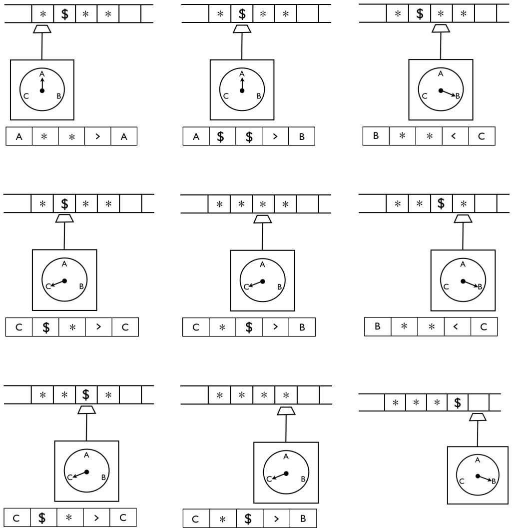
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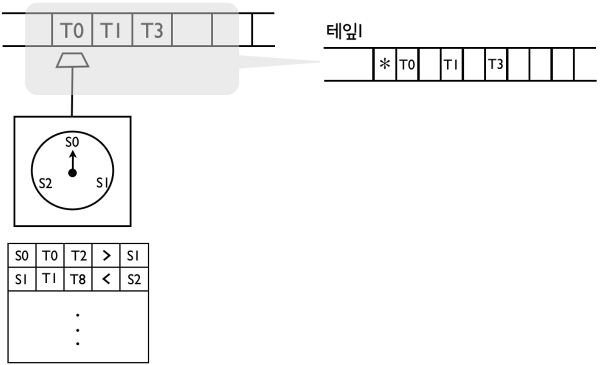
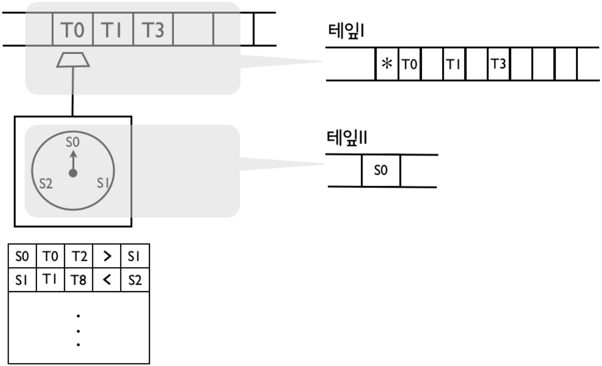
Expressing a TM in a tape (1/3)

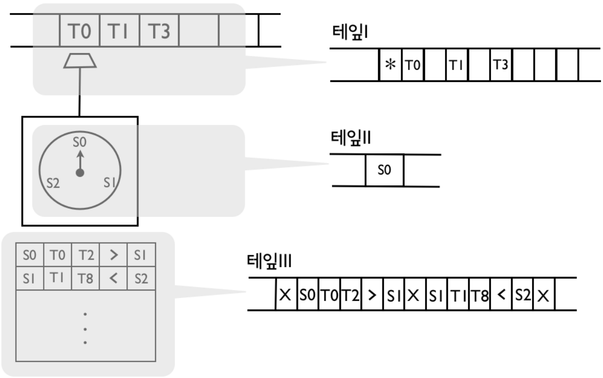


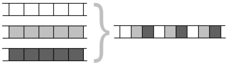
Expressing a TM in a tape (2/3)



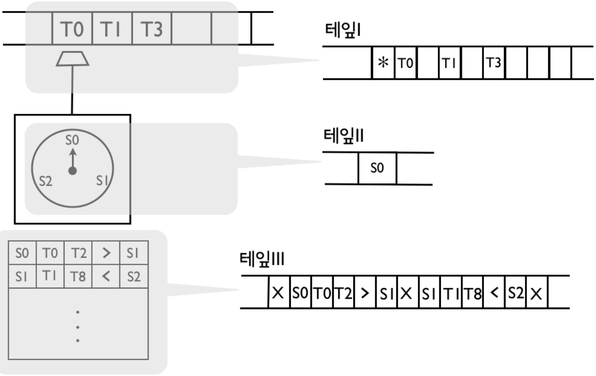
Universal TM: a Turing machine

Expressing a TM in a tape (3/3)





Universal TM: execution rules



- read tape-I, tape-II: $\mathtt{T}j$, $\mathtt{S}i$
- look for match in tape-III for $\mathtt{S}i$, $\mathtt{T}j$

$\mathtt{S}i$	$\mathtt{T}j$	$\mathtt{T}j'$	m	$\mathtt{S}i'$

- do as specified in the matched rule

$|\text{TMs}| \leq |\mathbb{N}|$

The number of TMs cannot be more than $|\mathbb{N}|$

- how many symbols for expressing TM? S,T,<,>,||, 0,$\cdots$,9, X,$*$
- a finite sequence: 17-ary number (a natural number)

Turing's Proof, by Universal TM and $|\text{TMs}| \leq |\mathbb{N}|$

- Lemma1 [$\exists$VERI $\implies \exists H$]. If a TM can generate all true formulas, then it can solve the halting problem.
- Lemma2 [$\nexists H$]. No TM can solve the halting problem.

Thus no TM can generate all true formulas.

Lemma1 proof: $\exists$VERI $\implies \exists H$

If a TM can generate all true formulas, then it can solve the halting problem.
Proof. $H(M) =$

1. run VERI by Universal TM
2. because VERI generates all true formulas, it generates either "M halts" or "not(M halts)."
3. answer accordingly. QED

Lemma2 proof: $\nexists H$

Proof. All TM and its inputs can be indexed by natural numbers: $M_1, M_2, \cdots, I_1, I_2, \cdots$.

1. If H exists, we can fill the following table.

	Input			
	I_1	I_2	I_3	$\cdots$
M_1	1	1	0	$\cdots$
M_2	1	0	1	$\cdots$
M_3	1	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

2. Then, following TM is different from all TMs!?

$$M(n) = \textit{Table}(M_n, I_n) \times U(M_n) + 1$$

Contradiction. Hence H is impossible. QED

Turing's 1936 Paper: Wrap up

"*On Computable Numbers, with an Application to the Entscheitungsproblem*"

1. define mechanical computation: turing machine
2. persuade us that TM is enough
3. assume machine VERI that generate all true formulas
4. show that machine VERI can solve the halting problem
5. prove that the halting problem is not computable

Hence VERI is not possible. QED

How Turing come up with the 1936 paper?

"Genius" Turing?

- What talents can generate "foundational knowledge"?
- Only "genius" can do that?
- Misleading message to students
- Not encouraging
- Maybe not true either

Did the 1936 paper come from an epiphany available only to "geniuses"?

Followings are my investigation on Turing's 1935

- Turing took Max Newman's class (Foundations of Mathematics and Gödel's Theorem) in 1935
- Turing learned about Gödel's Incompleteness proof there
- Turing was puzzled; why not more down-to-earth approach?
- Turing began his own style of the same proof

Given a 1st-order finite proof system about natural numbers, the incompleteness holds if such X exists as

$$X \text{ is not provable} = X$$

- X is either true or false.
- Suppose X is false, then X is provable.
  - inconsistent system, out of our discussion
- Suppose X is true, then X is not provable
  - only this is possible

$$X \text{ is not provable} = X$$

Is such X a 1st-order assertion about natural numbers? Gödel showed yes.

- unique natural numbers $\underline{f}$ and $\underline{p}$ for every 1st-order assertion f about natural numbers and every its proof tree p
- "X is not provable" is also an assertion about natural number: "$\underline{X}$ is a factor of a <u>proof</u>"

For $X = \mathit{UnProvable}(\underline{X})$

Gödel proved such X is

$$\begin{aligned} X &\stackrel{\text{def}}{=} G[x \mapsto k] \\ k &\stackrel{\text{def}}{=} \underline{G} \\ G &\stackrel{\text{def}}{=} \mathit{UnProvable}(\underline{\mathit{subst}(x, n, x)}) \quad (\text{Note } \mathit{fv}(G) = \{x\}) \\ & \quad \text{where } n = \underline{x} \text{ and } \mathit{subst}(a : \mathbb{N}, b : \mathbb{N}, c : \mathbb{N}) = \overline{a}[\overline{b} \mapsto c] \end{aligned}$$

because

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$$\begin{array}{rcl} X & = & G[x \mapsto k] \\ & = & (\mathit{UnProvable}(\underline{\mathit{subst}(x,n,x)}))[x \mapsto k] \\ & = & \mathit{UnProvable}(\underline{\mathit{subst}(k,n,k)}) \\ & = & \mathit{UnProvable}(\underline{G[x \mapsto k]}) \\ & = & \mathit{UnProvable}(\underline{X}) \qquad \text{QED} \end{array}$$

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- no nonsense: G has x replaced by itself(encoding k of G)
- two points from equation $X = \mathit{UnProvable}(X)$
  - an assertion about self is expressable within the given proof system
  - can interpret it as specifying an infinite object: a fixpoint of *UnProvable*: $X \stackrel{\text{def}}{=} \underbrace{\mathit{UnProvable}(\mathit{UnProvable}(\cdots))}_{\infty}$

$$
\begin{aligned}
x &= x + 1 \\
x &= x \multimap
\end{aligned}
$$

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- BTW, how a machine can see machines? A machine that has machines as its inputs?

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- Possibility expanded, now back to impossibility. I hope the halting problem is impossible. How can I prove it?

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- Hence $\nexists$VERI. QED

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