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A Polymorphic Modal Type System for Lisp-Like Multi-Staged Languages*

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Abstract

Our goal is to conservatively extend ML by all the Lisp's multi-staged programming constructs. The combination is meaningful because ML is a practical higher-order, impure, and typed language, while Lisp's macro system has long evolved complying with the demands from multi-staged programming practices.

This article presents a polymorphic modal type system and its principal type inference algorithm for ML + Lisp's staging constructs (the quasi-quote macro system). Our type system conservatively extends ML's let-polymorphism to support all features of Lisp's quasi-quote system. It supports open code, unrestricted operations on references, intensional variable-capturing substitution as well as capture-avoiding substitution, and lifting values into code. We believe our type system is the only one supporting such features with an inference algorithm.

1 Introduction

Staged computation, which explicitly divides a computation into separate stages, is a unifying framework for the existing program-generation systems. Partial evaluation [12, 5], runtime code generation [9, 18, 14, 15], function inlining, and macro expansion [21, 10] are all instances of staged computation. The stage levels can be arbitrarily large, determined by the nesting depth of program generations: stage 0 is for conventional non-staged programs, and a program of stage 0 generates a program of stage 1 that generates a program of stage 2, and so on.

The key aspect of multi-staged languages is to have code templates (program fragments) as first-class objects. Code templates are freely passed, stored, composed with other code of any stage, and executed when appropriate.

Our goal is to conservatively extend ML by all the Lisp's multi-staged programming constructs. The combination is meaningful because ML is a practical higher-order, impure, and typed language, while Lisp has long evolved complying with the demands from multi-staged programming practices.

This article presents a polymorphic type system and its principal type inference algorithm for ML + Lisp's staging constructs. Lisp (or Scheme) has the longest history of staged programming, and has evolved complying with the demands of macro programming practices. Lisp's staged programming features are all included in its so-called "quasi-quote macro" system. This macro system supports open code templates, imperative operations with code templates, intensional variable-capturing substitution (at the sacrifice of the alpha-equivalence) as well as capture-avoiding substitution (as "gensym" does) of free variables in open code templates, and lifting values into code templates. Our type system supports all these features.

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Contributions Our contributions are as follows.

- we present a polymorphic type system for a higher-order multi-staged language that supports all features of Lisp's quasi-quote macro programming:
  - open code: code with free variables can be constructed and composed without a restriction.
  - imperative operations with open code: open code can be stored, dereferenced, and overwritten without a restriction.
  - intensional variable-capturing substitution at stages > 0 ("unhygienic" macros): hence alpha-equivalence at stages > 0 (i.e., during macro definitions and expansions) is not preserved. This sacrifice, which may be unacceptable to a purely functional language, is a feature that Lisp's macro programmers have long enjoyed for efficiency and programming convenience.
  - capture-avoiding substitution ("hygienic" macros [13]) at stages > 0: the target language has an explicit new-name generation construct like Lisp's "gensym." Programmers use this construct to rename bound variables at runtime in order to avoid an unintensional variable-capture.
- our type system conservatively extends ML with the Lisp's quasi-quote macro system. ML's let-polymorphism with the value restriction is conservatively extended for imperative staged programs that handle open code templates as first-class objects. Also, ML's let-polymorphism is orthogonally combined with a record polymorphism that aims to allow a single open code template in multiple environments.
- we present the type system's principal type inference algorithm.

We believe our type system is only one supporting open code, unrestricted operations on references, both hygienic and unhygienic macros, and a type inference algorithm.

Comparisons Our type system differs from existing works as follows:

- Alpha-equivalence in our language is preserved only for stage 0 (non-staged) expressions; we enforce only closed code to be evaluated at stage 0. For higher stages the alpha equivalence is not preserved. If name change occurs in expressions of stages > 1, the result program's semantics becomes different from the original one.

  This non alpha-equivalence at stages > 1 (i.e.,"unhygienic macros") has been prevalent in macro programming, hence, we think, is worthwhile to be supported by our type system.
- For alpha-equivalence at stages > 0 (i.e., for "hygienic macros", or for staged programs that needs renaming at runtime), programmers can always choose to use the explicit new-name generation construct ("gensym" in Lisp, λ^* in our language). The construct $\lambda^* x.e$ uniquely renames bound variable x at runtime whenever a code is plugged inside e.
- For programs that are accepted by existing multi-staged type systems such as $\lambda^{\Box}$ [7, 8], $\lambda^{\bigcirc}$ [6] and λ^{i}[3] without the cross-stage persistence, there exist their semantics-preserving, translated versions that are accepted by our type system.
- Davies and Pfenning's system [7, 8] does not support open code templates.
- Nanevski and Pfenning's system[16, 17] supports open code, but does not support more than two stages and has no type inference algorithm.

- Calcagno et al.'s systems [3, 2] does not support imperative operations for open code templates. The environment classifiers [23, 3] allow restricted open code templates whose free variables' binders should be lexically visible at the same stage level.
- Chen and Xi's system [4] does not support free *named* variables inside code templates, hence it does not support hygienic manipulation of code templates.
- Ancona and Moggi's system [1] supports open code and imperative operations, but it supports only two-level stages, does not support intentional variable-capturing subsitution and has no type inference algorithm.

Examples Our type system's ideas and notable features in contrast to existing systems are illustrated with examples as follows. In examples, we use a mixed notation of Lisp's quasi-quote syntax [21] and ML-style expressions.

- Our type system allows open code templates irrespective of surrounding environments. Code template

  ```
  ‘(x+1)
  ```

  has type $\Box(\{x : int\}\rho \triangleright int)$, meaning that the code template can be executed or composed in any environment that has at least program variable `x` of integer type. $\{x : int\}\rho$ denotes a type environment that may have entries other than $\{x : int\}$. The postfix ρ is a *row variables* in record typing[20] that ranges over a finite set of field types.

- Our type system supports unrestricted imperative operations for open code templates. Our type system accepts the following program:

  ```
  let val a = ref ‘1
      val f = ‘(fn x -> ,(a := ‘(x + 1); ‘2))
   in ! a end
  ```

  where program variable `a` has type $\Box(\{x : int\}\rho \triangleright int)$ `ref`.

  This example is from [2] where it is untypable because their type system[2] restricts imperative operations only to closed code.

- Our type system supports unhygienic macros, where the programmers intensionally let free variables in macros be captured at runtime. Unhygienic macros are frequently used for both programming brevity and performance.

  Our type system, for example, accepts the following `map` function that generates a specialized code for the list argument:

  ```
  fun smap [] = ‘[]
    | smap (x::r) = ‘((f ,x) ::  ,(smap r))
  fun map ls = eval ‘(fn f => ,(smap ls))
  ```

  Note that the `smap` function generates open code whose free variable `f` will be bound after `smap`'s recursive calls. When using closed code only, we need as many extra closures and applications as the length of `ls`.

- Our type system supports hygienic macros [13] too that avoid the capture of free variables of open code templates during macro expansion. Suppose we define the `or` function as follows. The definition does not work as expected because of the variable-capture.

```
fun or a b =
   ‘(let val v = ,a in if v then v else ,b end)
```

When invoking `(or ‘false ‘`v`)`, we obtain an unintended code

```
‘(let val v = false in if v then v else v end).
```

Since the free variable v in the arguments is captured by v declared in the `or` function, the residual code returns `false` not argument v.

To avoid such unintended variable-capture during code composition we use a new language construct to generate fresh variable names: hygienic abstraction $\lambda^* v.e$ that substitutes a fresh name for v inside e before its application. Our type system accepts the following correct version of the `or` function:

```
fun or a b =‘((λ*v.if v then v else ,b) ,a)
```

Similarly, the `h2` function in Taha's thesis[22] can be defined and type-checked in our target language by using the hygienic abstraction λ^*:

```
fun h2 n z = if n=0 then z
     else ‘((λ*x.  ,(h2 (n-1) ‘(x + ,z))) n)
```

Note that call `(h2 2 ‘4)` evaluates to

```
‘((λx1.(λx2.x2+(x1+4)) 1) 2)
```

not to

```
‘((λx.(λx.x+(x+4)) 1) 2)
```

Note that `h2`'s unhygienic version is one where λx is used in stead of $\lambda^* x$. The programmer can choose between the two versions of `h2`, depending on his/her intension, and our type system accept both.

- Our type system uses record subtypes for the types of free variables in open code templates. For example,

```
if e then ‘(x+1) else ‘1
```

has type $\Box(\{x : int\}\rho \triangleright int)$ because the else-branch's type $\Box(\rho' \triangleright int)$ is a record subtype of the then-branch's type $\Box(\{x : int\}\rho \triangleright int)$ (the type operator $\triangleright$ is contravariant, like $\rightarrow$, over its left-hand side record type).

- Our type system orthogonally combines the let-polymorphism with open code templates. Consider

```
let val x = ‘y in ‘(,x + 1); ‘((,x 1) + z) end
```

Our type system assigns to `x` a polymorphic open code type $\forall\alpha\forall\rho.\Box(\{y : \alpha\}\rho\triangleright\alpha)$, which is distinctly instantiated for each occurrence of `x`. The first occurrence of `x` has $\Box(\{y : int\}\triangleright int)$ and the second has $\Box(\{y : int \rightarrow int, z : int\} \triangleright int \rightarrow int)$.

Organization Section 2 introduces notation. Section 3 presents a language λ^{sim}_{open} and its simple type system. Section 4 presents a polymorphic extension λ^{poly}_{open} of λ^{sim}_{open} and its polymorphic type system. Section 5 presents a principal type inference algorithm for λ^{poly}_{open}. Section 6 shows the relation of our type system to other existing ones. Section 7 concludes. Appendix has representative parts of the proofs for the lemmas in this article.

2 Notation

$A \oplus B$ is the union of disjoint sets A and B whereas $A \cup B$ is the union of arbitrary sets A and B. $[i..j]$ is the set of integers from i to j, and is empty if $i > j$. $\{x_i\}_{i=j}^{k}$ is $\{x_j, x_{j+1}, \ldots, x_k\}$, and is empty if $j > k$. $\{x_i\}_{i=j}^{k}$ is written $\{x_i\}_{j}^{k}$ without confusion.

Stage number is a non-negative integer. We use n for stage number. We assume that any expression for a stage number is always non-negative.

A relation R from set A to set B is any subset of the Cartesian product of A and B, i.e. $R \subseteq A \times B$. A relation $F \subseteq A \times B$ is a (partial) function if for all $a \in A$, $\{b \mid a : b \in F\}$ contains at most one element. The set of functions from set A to set B is denoted by $A \to B$. The set of functions from finite subsets of A to set B is denoted by $A \xrightarrow{fin} B$. The empty function is denoted by $\varnothing$. For a function F, $dom(F) = \{a \mid a : b \in F\}$ and $range(F) = \{b \mid a : b \in F\}$. $F|_A$ and $F|_A^-$ are the restrictions of a function F such that

$$F|_A = \{a : b \in F \mid a \in A\} \quad \text{and} \quad F|_A^- = \{a : b \in F \mid a \notin A\}.$$

$F_1 :: F_2$ is the union of domain-disjoint functions F_1 and F_2:

$$F_1 :: F_2 = \{a : b \mid a : b \in F_1 \text{ or } a : b \in F_2\},$$

where $dom(F_1) \cap dom(F_2) = \varnothing$. $F + a : b$ is the extension of a function F with $a : b$ such that $F + a : b = F|_{\{a\}}^- :: \{a : b\}$. $F_1 \cdots F_m$ is the sequence of functions $F_1, \ldots, F_m$. $F_1 \cdots F_m + a : b$ is the sequence of functions $F_1, \cdots, F_{m-1}, F_m + a : b$.

3 Simple-Typed Multi-Staged Language λ^{sim}_{open}

This section presents the multi-staged language λ^{sim}_{open} and its simple type system. λ^{sim}_{open} supports open code templates, unrestricted operations on references, both hygienic and unhygienic macros, and lifting values into code.

3.1 Syntax

$$\begin{array}{rcl} e \in Exp & ::= & c \mid x \mid \lambda x.e \mid e_1 e_2 \\ & & \mid \texttt{box}\, e \mid \texttt{unbox}_k\, e \mid \texttt{open}\, e \mid \texttt{lift}\, e \mid \lambda^* x.e \\ & & \mid \texttt{ref}\, e \mid \texttt{!}\, e \mid e_1 \texttt{:=} e_2 \\ & & \mid \texttt{clos}(x, e, \mathcal{E}) \mid l \end{array}$$

The syntax of λ^{sim}_{open} is based on the implicit $\lambda^{\Box}$ language [7, 8], and has additional expressions to manipulate code templates and to support imperative operations. Expression c is a constant of base types. Expressions $\texttt{box}\, e$ and $\texttt{unbox}_k\, e$ are for manipulating code templates, and respectively correspond to the backquote (‘), comma (,) ($k > 0$) and the `eval` ($k = 0$) in Lisp's quasi-quote notation. Expression $\texttt{open}\, e$ is useful for type inference by guiding the places of subtyping, and has no effect on evaluation. Expression $\texttt{lift}\, e$ is for converting the value of e into its corresponding code template. For hygienic code manipulation [13], expression $\lambda^* x.e$ renames x (of the same stage level) before a code is plugged inside e at runtime. Unless inside code templates, $\lambda^* x.e$ is not different from $\lambda x.e$. Expressions $\texttt{ref}\, e$, $\texttt{!}\, e$ and $e_1 \texttt{:=} e_2$ are

conventional for imperative operations. Expression $\texttt{clos}(x, e, \mathcal{E})$ is the closure of $\lambda x.e$ in an environment $\mathcal{E}$. Location l is for the store location. Closure and location, which are values, are included as expressions just for a convenience during our type system's soundness proofs.

3.2 Operational Semantics

Multi-staged language λ^{sim}_{open} has a call-by-value semantics. Figure 1 shows the big-step operational semantics. For brevity, we do not present error-generating rules, which are just complementary to Figure 1.

Evaluation

$$\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}')$$

means that an expression e under environment $\mathcal{E}$, store $\mathcal{S}$ and variables $\mathcal{V}$ evaluates to result r, store $\mathcal{S}'$ and variables $\mathcal{V}'$ at stage n.

Values are expressions not having a reducible expression. In multi-staged languages, values exist at every stage. Values at each stage are called staged values. A staged value v^n $(n > 0)$ is a frozen expression that is to be evaluated later when it is demoted to the stage 0 by the $\texttt{box}_0$ construct.

$$\begin{array}{lll}
\text{Staged Values} & v^n \in V^n & \\
\quad ::= & c \mid \texttt{box}\, v^1 \mid \texttt{clos}(x, e, \mathcal{E}) \mid l & \text{if } n = 0 \\
\quad ::= & c \mid x \mid \lambda x.v^n \mid v^n v^n & \\
\quad\quad \mid & \texttt{box}\, v^{n+1} \mid \texttt{unbox}_k\, v^{n-k} & \\
\quad\quad \mid & \texttt{open}\, v^n \mid \texttt{lift}\, v^n \mid \lambda^* x.v^n & \\
\quad\quad \mid & \texttt{ref}\, v^n \mid \texttt{!}\, v^n \mid v_1^n \texttt{:=} v_2^n & \\
\quad\quad \mid & \texttt{clos}(x, e, \mathcal{E}) \mid l & \text{if } n > k \geq 0
\end{array}$$

Variables	x, y, z, w	$\in$	*Var*	$=$	$\mathcal{V}_E \oplus \mathcal{V}_I$
Locations	l	$\in$	*Loc*	$=$	*a set of locations*
Stores	$\mathcal{S}$	$\in$	*Store*	$=$	$Loc \xrightarrow{fin} V^0$
Environments	$\mathcal{E}$	$\in$	*Env*	$=$	$Var \xrightarrow{fin} V^0$
Closures	$\texttt{clos}(x, e, \mathcal{E})$	$\in$	*Clos*	$=$	$Var \times Exp \times Env$
Results	r	$\in$	*Res*	$=$	$\bigcup_{n=0}^{\infty} V^n \oplus \{\texttt{err}\}$

Var is a countable set of variable names, and is the disjoint union of $\mathcal{V}_E$ and $\mathcal{V}_I$. $\mathcal{V}_E$ is a set of external variable names that may appear in the source program, while $\mathcal{V}_I$ is a set of internal variable names that $\lambda^* x.e$ may generate at runtime. The disjointness of $\mathcal{V}_I$ and $\mathcal{V}_E$ ensures that the internal variable names generated by $\lambda^* x.e$ are always different from the external variable names in the source program. $\mathcal{V}_I$ is simply denoted by $\mathcal{V}$ in the big-step operational semantics of λ^{sim}_{open}. We use x, y, z, w for variable names, and w for an internal variable name alone. Store $\mathcal{S}$ is a finite function from locations to values of stage 0. Environment $\mathcal{E}$ is a finite function from variables to values of stage 0. Expression $\texttt{clos}(x, e, \mathcal{E})$ is the closure of $\lambda x.e$ in an environment $\mathcal{E}$. Result r is either a value or $\texttt{err}$.

Lemma 3.1 *$V^n \subset V^{n+1}$ for $n \geq 0$.*

PROOF By induction on a staged value v^n in V^n. □

Lemma 3.2 *(Result) If $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}')$ then r is in $V^n \oplus \{\texttt{err}\}$.*

PROOF By induction on the type derivation of

$$\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}').$$

For a given evaluation, we proceed by cases on expression e. Lemma 3.1 is needed in cases of $\texttt{unbox}_k\, e$ and $\texttt{lift}\, e$ $(n = 0)$. □

(ECON) $$\mathcal{E},\mathcal{S},\mathcal{V} \vdash c \xrightarrow{n} (c,\mathcal{S},\mathcal{V})$$

(EVAR) $$\frac{x \in dom(\mathcal{E})}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash x \xrightarrow{0} (\mathcal{E}(x),\mathcal{S},\mathcal{V})} \qquad \mathcal{E},\mathcal{S},\mathcal{V} \vdash x \xrightarrow{n+1} (x,\mathcal{S},\mathcal{V})$$

(EABS) $$\mathcal{E},\mathcal{S},\mathcal{V} \vdash \lambda x.e \xrightarrow{0} (\mathtt{clos}(x,e,\mathcal{E}),\mathcal{S},\mathcal{V})$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{n+1} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \lambda x.e \xrightarrow{n+1} (\lambda x.v,\mathcal{S}',\mathcal{V}')}$$

(EAPP) $$\frac{\begin{array}{c}\mathcal{E},\mathcal{S},\mathcal{V} \vdash e_1 \xrightarrow{0} (\mathtt{clos}(x,e,\mathcal{E}'),\mathcal{S}_1,\mathcal{V}_1) \\ \mathcal{E},\mathcal{S}_1,\mathcal{V}_1 \vdash e_2 \xrightarrow{0} (v_2,\mathcal{S}_2,\mathcal{V}_2) \\ \mathcal{E}' + x : v_2,\mathcal{S}_2,\mathcal{V}_2 \vdash e \xrightarrow{0} (v_3,\mathcal{S}_3,\mathcal{V}_3)\end{array}}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e_1 e_2 \xrightarrow{0} (v_3,\mathcal{S}_3,\mathcal{V}_3)}$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e_1 \xrightarrow{n+1} (v_1,\mathcal{S}_1,\mathcal{V}_1) \quad \mathcal{E},\mathcal{S}_1,\mathcal{V}_1 \vdash e_2 \xrightarrow{n+1} (v_2,\mathcal{S}_2,\mathcal{V}_2)}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e_1 e_2 \xrightarrow{n+1} (v_1 v_2,\mathcal{S}_2,\mathcal{V}_2)}$$

(EBOX) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{n+1} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{box}\ e \xrightarrow{n} (\mathtt{box}\ v,\mathcal{S}',\mathcal{V}')}$$

(EEVAL) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{0} (\mathtt{box}\ v^1,\mathcal{S}_1,\mathcal{V}_1) \quad \mathcal{E},\mathcal{S}_1,\mathcal{V}_1 \vdash v^1 \xrightarrow{0} (v^0,\mathcal{S}_2,\mathcal{V}_2)}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{unbox}_0\ e \xrightarrow{0} (v^0,\mathcal{S}_2,\mathcal{V}_2)}$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{n+1} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{unbox}_0\ e \xrightarrow{n+1} (\mathtt{unbox}_0\ v,\mathcal{S}',\mathcal{V}')}$$

(EUNBOX) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{0} (\mathtt{box}\ v,\mathcal{S}',\mathcal{V}') \qquad k > 0}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{unbox}_k\ e \xrightarrow{k} (v,\mathcal{S}',\mathcal{V}')}$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{n+1} (v,\mathcal{S}',\mathcal{V}') \qquad k > 0}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{unbox}_k\ e \xrightarrow{n+1+k} (\mathtt{unbox}_k\ v,\mathcal{S}',\mathcal{V}')}$$

(EOPEN) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{0} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{open}\ e \xrightarrow{0} (v,\mathcal{S}',\mathcal{V}')}$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{n+1} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{open}\ e \xrightarrow{n+1} (\mathtt{open}\ v,\mathcal{S}',\mathcal{V}')}$$

(ELIFT) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{0} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{lift}\ e \xrightarrow{0} (\mathtt{box}\ v,\mathcal{S}',\mathcal{V}')}$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{n+1} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{lift}\ e \xrightarrow{n+1} (\mathtt{lift}\ v,\mathcal{S}',\mathcal{V}')}$$

(EGENSYM) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \lambda w.([x^n \overset{n}{\mapsto} w]\ e) \xrightarrow{n} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \oplus \{w\} \vdash \lambda^* x.e \xrightarrow{n} (v,\mathcal{S}',\mathcal{V}')}$$

(EREF) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{0} (v,\mathcal{S}',\mathcal{V}') \quad l \notin dom(\mathcal{S}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{ref}\ e \xrightarrow{0} (l,\mathcal{S}' + l : v,\mathcal{V}')}$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{n+1} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{ref}\ e \xrightarrow{n+1} (\mathtt{ref}\ v,\mathcal{S}',\mathcal{V}')}$$

(EDEREF) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{0} (l,\mathcal{S}',\mathcal{V}') \quad l \in dom(\mathcal{S}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash\ !\ e \xrightarrow{0} (\mathcal{S}'(l),\mathcal{S}',\mathcal{V}')}$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e \xrightarrow{n+1} (v,\mathcal{S}',\mathcal{V}')}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash\ !\ e \xrightarrow{n+1} (!\ v,\mathcal{S}',\mathcal{V}')}$$

(EASSIGN) $$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e_1 \xrightarrow{0} (l,\mathcal{S}_1,\mathcal{V}_1) \quad \mathcal{E},\mathcal{S}_1,\mathcal{V}_1 \vdash e_2 \xrightarrow{0} (v,\mathcal{S}_2,\mathcal{V}_2)}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e_1 := e_2 \xrightarrow{0} (v,\mathcal{S}_2 + l : v,\mathcal{V}_2)}$$

$$\frac{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e_1 \xrightarrow{n+1} (v_1,\mathcal{S}_1,\mathcal{V}_1) \quad \mathcal{E},\mathcal{S}_1,\mathcal{V}_1 \vdash e_2 \xrightarrow{n+1} (v_2,\mathcal{S}_2,\mathcal{V}_2)}{\mathcal{E},\mathcal{S},\mathcal{V} \vdash e_1 := e_2 \xrightarrow{n+1} (v_1 := v_2,\mathcal{S}_2,\mathcal{V}_2)}$$

(ELOC) $$\mathcal{E},\mathcal{S},\mathcal{V} \vdash l \xrightarrow{n} (l,\mathcal{S},\mathcal{V})$$

(ECLOS) $$\mathcal{E},\mathcal{S},\mathcal{V} \vdash \mathtt{clos}(x,e,\mathcal{E}') \xrightarrow{n} (\mathtt{clos}(x,e,\mathcal{E}'),\mathcal{S},\mathcal{V})$$

Figure 1: Big-step operational semantics of λ^{sim}_{open}: $\mathcal{V}$ means $\mathcal{V}_I$.

$$
\begin{array}{lll}
[x^n \overset{m}{\mapsto} w]\, c & = c & \\
[x^n \overset{m}{\mapsto} w]\, y & = w, & \text{if } x = y \text{ and } m = n \\
 & = y, & \text{otherwise} \\
[x^n \overset{m}{\mapsto} w]\, \lambda y.e & = \lambda y.e, & \text{if } x = y \text{ and } m = n \\
 & = \lambda y.([x^n \overset{m}{\mapsto} w]\, e), & \text{otherwise} \\
[x^n \overset{m}{\mapsto} w]\, (e_1 e_2) & = ([x^n \overset{m}{\mapsto} w]\, e_1)\, ([x^n \overset{m}{\mapsto} w]\, e_2) & \\
[x^n \overset{m}{\mapsto} w]\, \texttt{box}\, e & = \texttt{box}\, ([x^n \overset{m+1}{\mapsto} w]\, e) & \\
[x^n \overset{m}{\mapsto} w]\, \texttt{unbox}_k\, e & = \texttt{unbox}_k\, ([x^n \overset{m-k}{\mapsto} w]\, e) & \\
[x^n \overset{m}{\mapsto} w]\, \texttt{open}\, e & = \texttt{open}\, ([x^n \overset{m}{\mapsto} w]\, e) & \\
[x^n \overset{m}{\mapsto} w]\, \texttt{lift}\, e & = \texttt{lift}\, ([x^n \overset{m}{\mapsto} w]\, e) & \\
[x^n \overset{m}{\mapsto} w]\, \lambda^* y, e & = \lambda^* y.e, & \text{if } x = y \text{ and } m = n \\
 & = \lambda^* y.([x^n \overset{m}{\mapsto} w]\, e), & \text{otherwise} \\
[x^n \overset{m}{\mapsto} w]\, \texttt{ref}\, e & = \texttt{ref}\, ([x^n \overset{m}{\mapsto} w]\, e) & \\
[x^n \overset{m}{\mapsto} w]\, (\texttt{!}\, e) & = \texttt{!}\, ([x^n \overset{m}{\mapsto} w]\, e) & \\
[x^n \overset{m}{\mapsto} w]\, (e_1 \texttt{:=} e_2) & = ([x^n \overset{m}{\mapsto} w]\, e_1) \texttt{:=} ([x^n \overset{m}{\mapsto} w]\, e_2) & \\
[x^n \overset{m}{\mapsto} w]\, \texttt{clos}(y, e, \mathcal{E}) & = \texttt{clos}(y, e, \mathcal{E}), & \text{if } x = y \text{ and } m = n \\
 & = \texttt{clos}(y, [x^n \overset{m}{\rightarrow} w]\, e, \mathcal{E}), & \text{otherwise} \\
[x^n \overset{m}{\mapsto} w]\, l & = l &
\end{array}
$$

Figure 2: Substituting w for free variable x of stage n at stage m

Multi-staged language λ^{sim}_{open} extends the traditional lambda calculus conservatively. At stage 0, `(ECON)`, `(EVAR)`, `(EABS)` and `(EAPP)` are exactly the same as the call-by-value semantics of lambda calculus. The alpha-conversion and beta-reduction are available whenever necessary at stage 0. `(EREF)`, `(EDEREF)` and `(EASSIGN)` are for usual imperative operations at stage 0.

λ^{sim}_{open} can construct, compose and evaluate code templates at runtime. `(EBOX)` constructs a code template. λ^{sim}_{open} allows open expressions inside a code template. Stage number increases by one as we go inside a code template. If stage number is positive, we are inside a code template. `(EUNBOX)` merges code templates at stage $n - k$ into the current code at stage n. `(EUNBOX)` provides a way to escape from a code template temporarily before the code template ends. At stage 0, `(EEVAL)` converts a code template $\texttt{box}v^1$ into expression v^1 and then evaluates v^1, which is restricted to closed code using type system (See Subsection 3.3). No variable in code templates is bound to variable of the stage 0 since v^1 is always closed in `(EEVAL)`.

λ^{sim}_{open} forbids the implicit alpha-conversion inside code templates. To capture free variables at code composition, we need to preserve the names of free variables inside code templates. The meaning of an expression may change by the alpha-conversion inside code templates. Meanwhile, there is no restriction on the alpha-conversion at stage 0; as only closed code are evaluated in `(EEVAL)`, there is no way to bind variables in code templates to variables of stage 0. For example, $\lambda x.\texttt{unbox}_1\, y$ and $\lambda z.\texttt{unbox}_1\, y$ are alpha congruent at stage 1, but their meanings are different:

$$\{y : \texttt{box}\, x\} \vdash \lambda x.\texttt{unbox}_1\, y \xrightarrow{1} \lambda x.x,$$

whereas

$$\{y : \texttt{box}\, x\} \vdash \lambda z.\texttt{unbox}_1\, y \xrightarrow{1} \lambda z.x.$$

Meanwhile, $\lambda x.\texttt{unbox}_0 y$ and $\lambda z.\texttt{unbox}_0 y$ are alpha congruent at stage 0, and they have the same meaning since y is closed code.

λ^{sim}_{open} permits the explicit alpha-conversion inside code templates to manipulate code templates hygienically [13]. `(EGENSYM)` substitutes a fresh variable w for variable x in $\lambda^* x.e$ at stage n, and then evaluates the renamed lambda expression. At stage 0, $\lambda^* x.e$ has no effect

on evaluation since it is just an alpha-conversion in normal lambda calculus. For example, the following two expressions evaluate to different results:

$$\{y : \texttt{box}\, x\} \vdash \lambda x.\texttt{unbox}_1\ y \xrightarrow{1} \lambda x.x,$$

whereas

$$\{y : \texttt{box}\, x\} \vdash \lambda^* x.\texttt{unbox}_1\ y \xrightarrow{1} \lambda w.x$$

for some fresh internal variable w. Figure 2 shows the definition of the staged renaming operator $[x^n \overset{m}{\mapsto} w]$. In `(EGENSYM)`, x that is only at stage n should be replaced with w because the same "x" in some expression are not always the same variable. For example, at stage n, $\lambda x.\texttt{box}\, x$ has two different "x" variables; x in $\texttt{box}\, x$ is a variable at stage $n+1$, and is not bound with the lambda's x.

λ^{sim}_{open} has additional features to manipulate code templates. `(ELIFT)` lifts values to corresponding code templates. `(ELIFT)` is the reverse of `(EEVAL)`. Note that a staged value v^n can also be v^{n+1} (Lemma 3.1). `(EOPEN)` has no effect on evaluation. The `open` construct is a syntactic marker to which our type inference algorithm applies subtyping (See Section 3.3). Inside a code template, `(EABS)` reduces the body code in a lambda abstraction. Lambda expressions are not values but expressions inside a code template.

3.3 Type System

The key idea of the type system of λ^{sim}_{open} is to include a type environment in a code type. Type environments inside code types enable us to type open code templates, avoiding the restriction of Calcagno et al.'s systems[3, 2] that the types of free variables (even inside code values) are always looked up in the current type environment. Our type system supports imperative operations for open code templates, and both hygienic and unhygienic macros.

Types	A, B	$\in$	$Type$		
Type Environments	Γ	$\in$	$TyEnv$	$=$	$Var \overset{fin}{\rightarrow} Type$
Store Typings	Σ	$\in$	ST	$=$	$Loc \overset{fin}{\rightarrow} Type$

Figure 3 shows a type system for λ^{sim}_{open} in the style of Harper [11]; all type rules need store typing in order to support imperative operations. We use A, B for types. Type environment Γ is a finite function from variables to types. Store typing Σ is a finite function from locations to types.

$$\text{Types} \qquad A, B \quad ::= \quad \iota \mid A \to B \mid \Box(\Gamma \triangleright A) \mid A\ \texttt{ref}$$

We use ι for base type, $A \to B$ for function types, and $\Box(\Gamma \triangleright A)$ for code template types. $\Box(\Gamma \triangleright A)$ is a conditional modal type in which the condition Γ specifies the types of free program variables in the code template of type A. We use $A\,\texttt{ref}$ for types of store locations having A-typed values.

Definition 3.1 *A store $\mathcal{S}$ is well typed with respect to a store typing Σ, written $\models \mathcal{S} : \Sigma$, if and only if $dom(\mathcal{S}) = dom(\Sigma)$ and $\Sigma; \varnothing \vdash \mathcal{S}(l) : \Sigma(l)$ for every $l \in dom(\mathcal{S})$.*

Definition 3.2 *An environment $\mathcal{E}$ is well typed with respect to a store typing Σ and a type environment Γ, written $\Sigma \models \mathcal{E} : \Gamma$, if and only if $dom(\mathcal{E}) = dom(\Gamma)$ and $\Sigma; \varnothing \vdash \mathcal{E}(x) : \Gamma(x)$ for every $x \in dom(\mathcal{E})$.*

$$\texttt{(TSCON)} \qquad \Sigma;\Gamma_0\cdots\Gamma_n \vdash c : \iota$$

$$\texttt{(TSVAR)} \qquad \frac{\Gamma_n(x) = A}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash x : A}$$

$$\texttt{(TSABS)} \qquad \frac{\Gamma_0\cdots\Gamma_n + x : A \vdash e : B}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \lambda x.e : A \to B}$$

$$\texttt{(TSAPP)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e_1 : A \to B \qquad \Sigma;\Gamma_0\cdots\Gamma_n \vdash e_2 : A}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e_1 e_2 : B}$$

$$\texttt{(TSBOX)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n\Gamma \vdash e : A}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \texttt{box}\, e : \Box(\Gamma \triangleright A)}$$

$$\texttt{(TSUNBOX)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e : \Box(\Gamma_{n+k} \triangleright A) \qquad k > 0}{\Sigma;\Gamma_0\cdots\Gamma_n\cdots\Gamma_{n+k} \vdash \texttt{unbox}_k e : A}$$

$$\texttt{(TSEVAL)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e : \Box(\varnothing \triangleright A)}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \texttt{unbox}_0\, e : A}$$

$$\texttt{(TSLIFT)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e : A}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \texttt{lift}\, e : \Box(\Gamma \triangleright A)}$$

$$\texttt{(TSGENSYM)} \qquad \frac{\begin{array}{c}\Sigma;\Gamma_0\cdots\Gamma_n + w : A \vdash [x^n \overset{n}{\mapsto} w]\, e : B \\ w \text{ is a fresh prog. var. in } (\Sigma, \Gamma_0\cdots\Gamma_n, \lambda^* x.e)\end{array}}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \lambda^* x.e : A \to B}$$

$$\texttt{(TSOPEN)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e : \Box(\varnothing \triangleright A)}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \texttt{open}\, e : \Box(\Gamma \triangleright A)}$$

$$\texttt{(TSREF)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e : A}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \texttt{ref}\, e : A\,\texttt{ref}}$$

$$\texttt{(TSDEREF)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e : A\,\texttt{ref}}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \texttt{!}\, e : A}$$

$$\texttt{(TSASSIGN)} \qquad \frac{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e_1 : A\,\texttt{ref} \qquad \Sigma;\Gamma_0\cdots\Gamma_n \vdash e_2 : A}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash e_1 \texttt{:=} e_2 : A}$$

$$\texttt{(TSLOC)} \qquad \frac{\Sigma(l) = A}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash l : A\,\texttt{ref}}$$

$$\texttt{(TSCLOS)} \qquad \frac{\Sigma \models \mathcal{E} : \Gamma \qquad \Sigma;\Gamma + x : A \vdash e : B}{\Sigma;\Gamma_0\cdots\Gamma_n \vdash \texttt{clos}(x, e, \mathcal{E}) : A \to B}$$

Figure 3: Type system for λ^{sim}_{open}

Typing judgment

$$\Sigma;\Gamma_0\cdots\Gamma_n \vdash e : A$$

means that an expression e, under store typing Σ and type environments $\Gamma_0\cdots\Gamma_n$, has type A at stage n. $\Gamma_0\cdots\Gamma_n$ is a sequence of type environments $\Gamma_0, \ldots, \Gamma_n$. Γ_n is the current type environment. Subscripts $0, \ldots, n$ are stage numbers.

(TSBOX) makes open code template box e typable since

$$\Gamma_0\cdots\Gamma_n\Gamma \vdash e : A$$

may hold for an open expression e. Type $\Box(\Gamma \triangleright A)$ is for code template box e, indicating that e has type A in all accessible stages satisfying Γ. Note that code template type has a subtype property: if $\Gamma_0\cdots\Gamma_n \vdash \texttt{box}\, e : \Box(\Gamma \triangleright A)$ holds, then $\Gamma_0\cdots\Gamma_n \vdash \texttt{box}\, e : \Box(\Gamma' \triangleright A)$ also holds for $\Gamma' \supseteq \Gamma$. (TSUNBOX) checks if a code template from the previous stage is properly captured

by the type environment at the current stage. `(TSUNBOX)` shows that code template types have modal property: if e has type $\Box(\Gamma \triangleright A)$ at stage Γ_n, then $\texttt{unbox}_k\ e$ has type A in an accessible stage satisfying Γ. `(TSEVAL)` allows only closed code to be evaluated by the `eval` construct. Type $\Box(\varnothing \triangleright A)$ is the same as type $\Box A$ in Davies and Pfenning [7, 8]. `(TSOPEN)` relaxes closed code types to behave as open ones. Expression `open` e induces a syntax-driven subtyping. Closed code type $\Box(\varnothing \triangleright A)$ can be relaxed to $\Box(\Gamma \triangleright A)$ if indicated by the `open` construct. Such relaxation not only expands the set of typable expressions but also enables a syntax-driven principal type inference algorithm (See Section 5).

`(TSGENSYM)` says that the type of $\lambda^* x.e$ is the same as the type of the expression resulting from renaming its bound variable x by a fresh program variable w. `(TSGENSYM)` requires the explicit alpha-conversion hence the type of $\lambda^* x.e$ can be different from that of $\lambda x.e$. For example,

$$\{y : \Box(\{x : int\} \triangleright int)\}\ \ \{x : int\} \vdash \lambda x.\texttt{unbox}_1\ y : int \to int,$$
$$\{y : \Box(\{x : int\} \triangleright int)\}\ \ \{x : int\} \vdash \lambda^* x.\texttt{unbox}_1\ y : A \to int \ \text{ for some } A.$$

Hence,

$$\{y : \Box(\{x : int\} \triangleright int)\}\ \ \{x : int\} \vdash (\lambda x.\texttt{unbox}_1\ y)\ true \quad \text{is untypable but}$$
$$\{y : \Box(\{x : int\} \triangleright int)\}\ \ \{x : int\} \vdash (\lambda^* x.\texttt{unbox}_1\ y)\ true : int.$$

`(TSLIFT)` allows typing the code template that corresponds to the value of e. Lemma 3.1 shows that a value of stage n is also a value of stage $n+1$. Hence, if e has type A, then e's value is also an A-typed value at a higher stage under any context, hence $\Box(\Gamma \triangleright A)$ for an arbitrary Γ.

Unlike the type system of Calcagno et al. [2], `(TSREF)`, `(TSDEREF)` and `(TSASSIGN)` support imperative operations for open code templates without any restriction. `(TSCON)`, `(TSVAR)`, `(TSABS)` and `(TSAPP)` are as usual except for extending to multi-staged setting.

3.4 Soundness of Type System

First, consider `(EEVAL)` at stage 0. `(EEVAL)` converts $\texttt{box}\, v^1$ into v^1 at stage 0. By the following demotion lemma, if

$$\Sigma; \varnothing \vdash \texttt{box}\ v^1 : \Box(\Gamma \triangleright A)$$

then $\Sigma; \Gamma \vdash v^1 : A$ because $\Sigma; \varnothing\Gamma \vdash v^1 : A$ by `(TSBOX)`. Hence, code template of type $\Box(\Gamma \triangleright A)$ can be safely converted into expressions of type A under type environment Γ.

Lemma 3.3 *(Demotion) If* $\Sigma; \varnothing\Gamma_1 \cdots \Gamma_n \vdash v : A$ *for* $n > 0$, *then* $\Sigma; \Gamma_1 \cdots \Gamma_n \vdash v : A$.

Proof By induction on the type derivation of $\Sigma; \varnothing\Gamma_1 \cdots \Gamma_n \vdash v : A$. For a given type derivation, we proceed by cases on the finally used type rule. □

Second, a well typed expression preserves its type after evaluation in λ^{sim}_{open}. For the proof of the preservation lemma, we need the result lemma (Lemma 3.2) and the demotion lemma (Lemma 3.3).

Lemma 3.4 *(Preservation) If* $\models \mathcal{S} : \Sigma$, $\Sigma \models \mathcal{E} : \Gamma_0$,

$$\Sigma; \Gamma_0 \cdots \Gamma_n \vdash e : A \ \text{ and } \ \mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}'),$$

then $\Sigma'; \varnothing\Gamma_1 \cdots \Gamma_n \vdash r : A$ *and* $\models \mathcal{S}' : \Sigma'$ *for some* $\Sigma' \supseteq \Sigma$.

Proof By induction on the type derivation $\Sigma; \Gamma_0 \cdots \Gamma_n \vdash e : A$ and evaluation $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}')$. For a given type derivation, we proceed by cases on the finally used type rule. □

Finally, the type system of λ^{sim}_{open} is sound. If a closed expression is well typed, then it preserves the type after evaluation. Hence, the evaluation result can not be `err` because `err` is not typable.

$$
\begin{array}{lll}
[x^n \overset{m}{\mapsto} w]\ \texttt{let}\ (y\ e_1)\ e_2 & & \\
\quad = \quad \texttt{let}\ (y\ \ ([x^n \overset{m}{\mapsto} w]\ e_1))\ \ e_2, & & \text{if } x = y \text{ and } m = n \\
\quad = \quad \texttt{let}\ (y\ \ ([x^n \overset{m}{\mapsto} w]\ e_1))\ \ [x^n \overset{m}{\mapsto} w]\ e_2, & & \text{otherwise} \\
\\
\textit{Other cases} & \textit{The same as } \lambda^{\text{sim}}_{\text{open}} \textit{ in Figure 2} &
\end{array}
$$

Figure 4: Substitution at stage m of w for free variable x declared at stage n

$$
\begin{array}{ll}
\texttt{(ELET)} & \dfrac{\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{0} (v_1, \mathcal{S}_1, \mathcal{V}_1) \qquad \mathcal{E} + x : v_1, \mathcal{S}_1, \mathcal{V}_1 \vdash e_2 \xrightarrow{0} (v_2, \mathcal{S}_2, \mathcal{V}_2)}{\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{let}\ (x\ e_1)\ e_2 \xrightarrow{0} (v_2, \mathcal{S}_2, \mathcal{V}_2)} \\
\\
 & \dfrac{\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{n+1} (v_1, \mathcal{S}_1, \mathcal{V}_1) \qquad \mathcal{E}, \mathcal{S}_1, \mathcal{V}_1 \vdash e_2 \xrightarrow{n+1} (v_2, \mathcal{S}_2, \mathcal{V}_2)}{\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{let}\ (x\ e_1)\ e_2 \xrightarrow{n+1} (\texttt{let}\ (x\ v_1)\ v_2, \mathcal{S}_2, \mathcal{V}_2)} \\
\\
\textit{Other rules} & \textit{The same as } \lambda^{\text{sim}}_{\text{open}} \textit{ in Figure 1}
\end{array}
$$

Figure 5: Big-step operational semantics of λ^{poly}_{open}: $\mathcal{V}$ is a set of internal program variables $\mathcal{V}_I$.

Theorem 3.1 *(Soundness) If*

$$\varnothing; \varnothing \vdash e : A \quad \textit{and} \quad \varnothing, \varnothing, \mathcal{V} \vdash e \xrightarrow{0} (r, \mathcal{S}, \mathcal{V}'),$$

then $\Sigma; \varnothing \vdash r : A$ *where* $\models \mathcal{S} : \Sigma$.

Proof The empty store is well typed with respect to the empty store typing: $\models \varnothing : \varnothing$. The empty environment is also well typed with respect to the empty type environment and the empty store typing: $\varnothing \models \varnothing : \varnothing$. From the assumptions and the preservation lemma, it follows that $\Sigma; \varnothing \vdash r : A$ where $\models \mathcal{S} : \Sigma$. □

4 Polymorphic Multi-Staged Language λ^{poly}_{open}

This section presents a let-polymorphic multi-staged language λ^{poly}_{open} and its polymorphic type system. The type system adopts the let-polymorphism where type generalization occurs only in `let` expressions. This polymorphic generalization is applied also to the record types that are used for the type environments accompanying code types. The record types are generalized in a different axis too, in the sense of record subtyping in order to relax the constraints imposed by the type environments. Syntactic value restriction is used to support imperative operations in the polymorphic type system.

4.1 Syntax

λ^{poly}_{open} extends λ^{sim}_{open} by the let-binding expression $\texttt{let}\ (x\ e_1)\ e_2$ that binds the value of e_1 to x in e_2.

$$
\begin{array}{rcl}
e \in \mathit{Exp} & ::= & c \mid x \mid \lambda x.e \mid e_1 e_2 \mid \texttt{let}\ (x\ e_1)\ e_2 \\
 & \mid & \texttt{box}\, e \mid \texttt{unbox}_k\, e \mid \texttt{open}\, e \mid \texttt{lift}\, e \mid \lambda^* x.e \\
 & \mid & \texttt{ref}\, e \mid \texttt{!}\, e \mid e_1 \texttt{:=} e_2 \\
 & \mid & \texttt{clos}(x, e, \mathcal{E}) \mid l
\end{array}
$$

4.2 Operational Semantics

Figure 5 shows the big-step operational semantics of $\texttt{let}\ (x\ e_1)\ e_2$. Other evaluation rules are the same as those for λ^{sim}_{open}. The error generation rules are just complementary to the semantics of the normal evaluation.

$$
\begin{array}{lll}
\text{Staged Values } v^n \in V^n & & \\
\quad ::= & c \mid \texttt{box}\, v^1 \mid \texttt{clos}(x, e, \mathcal{E}) \mid l & \text{if } n = 0 \\
\quad ::= & c \mid x \mid \lambda x.v^n \mid v^n v^n & \\
\quad\quad \mid & \texttt{box}\, v^{n+1} \mid \texttt{unbox}_k\, v^{n-k} & \\
\quad\quad \mid & \texttt{open}\, v^n \mid \texttt{lift}\, v^n \mid \lambda^* x.v^n & \\
\quad\quad \mid & \texttt{ref}\, v^n \mid \texttt{!}\, v^n \mid v_1^n \texttt{:=} v_2^n & \\
\quad\quad \mid & \texttt{let}\, (x\ v^n)\ v^n & \\
\quad\quad \mid & \texttt{clos}(x, e, \mathcal{E}) \mid l & \text{if } n > k \geq 0
\end{array}
$$

λ^{poly}_{open} has $\texttt{let}\ (x\ v^n)\ v^n$ as additional staged values. Other staged values are the same as in λ^{sim}_{open}. Except for staged values, other semantic domains *Loc, Store, Var, Env, Clos* and *Res* are the same as in λ^{sim}_{open} (See Section 3.2).

Lemma 4.1 *$V^n \subset V^{n+1}$ for $n \geq 0$.*

Proof By induction on a staged value v^n in V^n. □

Lemma 4.2 *(Result) If $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}')$ then r is in $V^n \oplus \{\texttt{err}\}$.*

Proof By induction on the type derivation of

$$\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}').$$

For a given type derivation, we proceed by cases on expression e. □

4.3 Staged, Syntactic Value Restriction

Let-polymorphic generalization is activated only when expression e_1 in $\texttt{let}\ (x\ e_1)\ e_2$ never expands the store. We need to statically check whether an expression expands the store or not.

One thing we have to be careful in devising such a check is that a non-expansive expression at stage n should not expand the store during its evaluation at any stage $m \leq n$. This *demotion-closedness* is because any expression at stage n can be demoted by the $\texttt{unbox}_0$ construct to stage m for some $m < n$ until it evaluates to a value of stage 0.

Staged predicate expansive$^n(e)$ in Figure 6 is designed based on Wright [26]: if an expression is a syntactic value, then it never expands the store and hence it is a non-expansive expression. Note that expansive$^n(e)$ satisfies the *demotion-closedness*. Consider the cases of $\lambda x.e$ and $\lambda^* x.e$. If e is in V^1, then $\lambda x.e$ and $\lambda^* x.e$ are in V^1. Then, from Lemma 4.1, $\lambda x.e$ and $\lambda^* x.e$ are in V^n for any $n > 0$ (and hence they never expand the store at stage n). Meanwhile, at stage 0, $\lambda x.e$ and $\lambda^* x.e$ never expand the store; they just reduce to closures. Thus, if e is in V^1 or at stage 0, $\lambda x.e$ and $\lambda^* x.e$ never expand the store for any stage n. The $\texttt{box}\, e$ case is similar as follows. If $e \in V^1$, then, $\texttt{box}\, e \in V^0$ by definition of V^n. By Lemma 4.1, $\texttt{box}\, e \in V^0 \subset V^{n+1}$ for $n \geq 0$. Hence, $\texttt{box}\, e$ is a value at any stage; it is a non-expansive expression. $\texttt{ref}\, e$ expands the store at stage 0. Hence, expansive$^n(\texttt{ref}\, e)$ is defined to be true for any stage in order to satisfy *demotion-closedness*. $e_1 e_2$ may expand the store at stage 0. Hence, expansive$^n(e_1 e_2)$ is also defined to be true for any stage. Expression $\texttt{unbox}_k\, e$ may expand the store at stage k. Hence, expansive$^n(\texttt{unbox}_k e)$ is true for any stage $n \geq k$. At stage $n < k$, $\texttt{unbox}_k\, e$ is meaningless. Expressions $\texttt{!}\, e_1$, $e_1 \texttt{:=} e_2$, $\texttt{open}\, e_1$, $\texttt{lift}\, e_1$ and $\texttt{let}\ (x\ e_1)\ e_2$ expand the store if e_1 or e_2 expands the store. Expressions c, x $\texttt{clos}(x, e, \mathcal{E})$ and l do not expand the store for any stage. Note that expansive$^n(\texttt{clos}(x, e, \mathcal{E}))$ and expansive$^n(l)$ are not used in type inference because they are values not occurring in the source program. They are included just for our proofs.

$$
\begin{array}{lcll}
\text{expansive}^n(c) & = & \mathit{False} & \\
\text{expansive}^n(x) & = & \mathit{False} & \\
\text{expansive}^n(\lambda x.e) & = & \mathit{False}, & \text{if } n = 0 \vee e \in V^1 \\
 & = & \mathit{True}, & \text{otherwise} \\
\text{expansive}^n(e_1 e_2) & = & \mathit{True} & \\
\text{expansive}^n(\texttt{box}\ e) & = & \mathit{False}, & \text{if } e \in V^1 \\
 & = & \mathit{True}, & \text{otherwise} \\
\text{expansive}^n(\texttt{unbox}_k\ e) & = & \mathit{True}, & n \geq k \\
\text{expansive}^n(\texttt{open}\ e) & = & \text{expansive}^n(e) & \\
\text{expansive}^n(\texttt{lift}\ e) & = & \text{expansive}^n(e) & \\
\text{expansive}^n(\lambda^* x.e) & = & \mathit{False}, & \text{if } n = 0 \vee e \in V^1 \\
 & = & \mathit{True}, & \text{otherwise} \\
\text{expansive}^n(\texttt{ref}\ e) & = & \mathit{True} & \\
\text{expansive}^n(\texttt{!}\ e) & = & \text{expansive}^n(e) & \\
\text{expansive}^n(e_1 \texttt{:=} e_2) & = & \text{expansive}^n(e_1) \vee \text{expansive}^n(e_2) & \\
\text{expansive}^n(\texttt{let}\ (x\ e_1)\ e_2) & = & \text{expansive}^n(e_1) \vee \text{expansive}^n(e_2) & \\
\text{expansive}^n(\texttt{clos}(x, e, \mathcal{E})) & = & \mathit{False} & \\
\text{expansive}^n(l) & = & \mathit{False} &
\end{array}
$$

Figure 6: Staged predicate $\text{expansive}^n(e)$: a non-expansive expression at stage n is syntactically guaranteed never to expand the store during its evaluation at stage m for every $m \leq n$.

4.4 Type System

Our polymorphic type system extends λ^{sim}_{open} by ML's let-polymorphism and Rémy's record type [20].

$$
\begin{array}{lcll}
\text{Types} & A, B & \in & \mathit{Type} \\
 & & ::= & \iota \mid A \to B \mid \Box(\Gamma \triangleright A) \mid A\ \texttt{ref} \mid \alpha \\
\text{Type Variables} & \alpha, \beta & \in & \mathit{TyVar} \\
\text{Fields} & F, G & \in & \mathit{Field} = \mathit{Type} \oplus \bot \\
\text{Field Variables} & \theta & \in & \mathit{FieldVar} \\
\text{Store Typings} & \Sigma & \in & \mathit{StoTy} = \mathit{Loc} \xrightarrow{\text{fin}} \mathit{Type} \\
\text{Type Environments} & \Gamma & \in & \mathit{TyEnv} = \mathit{Var} \xrightarrow{\text{fin}} \mathit{Field} \\
 & & & ::= \{x_i : F_i\}_1^m \mid \{x_i : F_i\}_1^k \rho_L \\
 & & & \quad \text{where } L = \{x_i\}_1^k \\
\text{Type Environments Variables} & \rho & \in & \mathit{TyEnvVar}
\end{array}
$$

We use A, B for types, and α, β for type variables. Field[20] is either Type or $\bot$. We use F, G for fields, and θ for field variables. Store typing Σ is a finite function from locations to types. Type environment Γ is a finite function from variables to fields, and can be regarded as a record. We use ρ for type environment variables. Type environment $\{x_i : F_i\}_1^m \rho_L$ denotes a type environment

$$\{x_i : F_i\}_1^m :: \{y_i : G_i\}_1^k \quad \text{where} \quad \{y_i\}_1^k \cap L = \varnothing$$

where :: is a domain-disjoint union. Operator :: extends for type environment variables as

$$\{x_i : F_i\}_1^m :: \rho = \{x_i : F_i\}_1^m \rho.$$

In an type environment, a field is a type or $\bot$. If $x : \bot$ in Γ, x is an undefined variable in Γ. (Such extension to $\bot$, following the idea in Rémy's record typing [20], is needed only for the type inference algorithm. See Section 5.) Hence,

$$\{x_i : A_i\}_1^k \quad \text{and} \quad \{x_i : A_i\}_1^k :: \{y_i : \bot\}_1^m \text{ for some } m$$

are the same type environment. Similarly,

$$\rho_L \quad \text{and} \quad \{x : \theta\} :: \rho_{L \oplus \{x\}}$$

are the same type environment since field variable θ can be either A or $\bot$. Without confusion, we write $\{x_i : F_i\}_1^k \rho$ instead of $\{x_i : F_i\}_1^k \rho_L$.

$$\begin{array}{lcl} \xi, \chi & \in & \mathit{TyVar} \oplus \mathit{TyEnvVar} \oplus \mathit{FieldVar} \\ X, Y & \in & \mathit{Type} \oplus \mathit{TyEnv} \oplus \mathit{Field} \end{array}$$

Some symbols range over multiple sets: ξ, χ are used for type variables, type environment variables, or field variables. X, Y are used for types, type environments, or fields.

$$\begin{array}{lrcll} \text{Type Schemes} & \tau, \sigma & \in & \mathit{TyScheme} & ::= \quad \forall \xi.\tau \mid A \\ \text{Field Schemes} & \mu, \pi & \in & \mathit{FieldScheme} & = \quad \mathit{TypeScheme} \oplus \bot \\ \text{Type Scheme Env} & \Delta & \in & \mathit{TySchemeEnv} & = \quad \mathit{Var} \xrightarrow{\text{fin}} \mathit{FieldScheme} \\ & & & & ::= \quad \{x_i : \mu_i\}_1^m \mid \{x_i : \mu_i\}_1^k \rho_L \\ & & & & \qquad \text{where } L = \{x_i\}_1^k \end{array}$$

We use τ, σ for type schemes, and μ, π for field schemes. Type schemes are in the prenex form, containing outermost quantification only. $\forall \xi.\tau$ binds ξ in τ. A type variable α, field variable θ or a type environment variable ρ is free in τ if it occurs in τ and is not bound. A type scheme $\forall \xi_1 \ldots \forall \xi_m.\sigma$ is written $\forall \xi_1 \ldots \xi_m.\sigma$. Type scheme environment is a finite function from variables to field schemes. Type scheme environment is an extension of type environment using field schemes. Similarly to type environments, type environment schemes may contain type environment variables. Type scheme $\{x_i : \mu_i\}_1^m \rho_L$ denotes some type scheme environment

$$\{x_i : \mu_i\}_1^m :: \{y_i : F_i\}_1^k \text{ where } \{y_i\}_1^k \cap L = \varnothing.$$

Note that a variable ρ is not for type scheme environments but for type environments. Type environments are defined to be monomorphic. Like type environments,

$$\{x_i : \tau_i\}_1^k \quad \text{and} \quad \{x_i : \tau_i\}_1^k :: \{y_i : \bot\}_1^m \text{ for some } m$$

are the same, and

$$\{x_i : \tau_i\}_1^k \rho_L \quad \text{and} \quad \{x_i : \tau_i\}_1^k :: \{y_i : \theta_i\}_1^m :: \rho_{L \oplus \{y_i\}_1^m}$$

are the same.

Definition 4.1 *(Free variables)* $\mathrm{FV}(\tau)$ *is the set of free type variables, free field variables and free type environment variables occurring in* τ. FV *is abused for type schemes, field schemes, store typings and type scheme environments:*

$$\mathrm{FV}(\Sigma) \;=\; \textstyle\bigcup_{A \in \mathrm{range}(\Sigma)} \mathrm{FV}(A)$$

and

$$\begin{array}{rcll} \mathrm{FV}(\Delta) & = & \bigcup_{i=1}^m \mathrm{FV}(\mu_i) \cup \{\rho\}, & \text{if } \Delta = \{x_i : \mu_i\}_1^m \rho \\ & = & \bigcup_{i=1}^m \mathrm{FV}(\mu_i), & \text{if } \Delta = \{x_i : \mu_i\}_1^m. \end{array}$$

Definition 4.2 *(Bound variables)* $\mathrm{BV}(\tau)$ *is the set of bound type variables, bound field variables and bound type environment variables occurring in* τ. BV *is abused for type schemes, field schemes and type scheme environments:*

$$\mathrm{BV}(\Delta) = \textstyle\bigcup_{i=1}^m \mathrm{BV}(\mu_i)$$

where $\Delta = \{x_i : \mu_i\}_1^m \rho$ *or* $\Delta = \{x_i : \mu_i\}_1^m$.

$$\begin{array}{l} \text{Substitutions } R, S, T, U \quad \in \quad \mathit{TySub} \\ \quad = (\mathit{FieldVar} \xrightarrow{\text{fin}} \mathit{Field}) :: (\mathit{TyVar} \xrightarrow{\text{fin}} \mathit{Type}) :: (\mathit{TyEnvVar} \xrightarrow{\text{fin}} \mathit{TyEnv}) \end{array}$$

A substitution is a finite function from field variables to fields, from type variables to types, and from type environment variables to type environments. Applying substitution R to X is written RX. The composition of substitutions S followed by R is written $R \cdot S$ or RS, and is defined as

$$\{\xi : R(S(\xi)) \mid \xi \in \mathit{dom}(S)\} :: \{\xi : R(\xi) \mid \xi \in \mathit{dom}(R) \setminus \mathit{dom}(S)\}.$$

Definition 4.3 *(Type instantiation) A type scheme $\forall\xi_1 \ldots \xi_m.A$ instantiates to a type B, written $\forall\xi_1 \ldots \xi_m.A \succ B$ if and only if there exists a substitution S with domain $\{\xi_i\}_1^m$ and $SA = B$.*

Definition 4.4 *(Type environment instantiation) A type scheme environment $\{x_i : \mu_i\}_1^m\rho$ instantiates to a type environment Γ, written $\{x_i : \mu_i\}_1^m\rho \succ \Gamma$, if and only if $\Gamma = \{x_i : F_i\}_1^m\rho$ and $\mu_i \succ F_i$ for $i \in [1..m]$. Likewise, $\{x_i : \mu_i\}_1^m$ instantiates to Γ if and only if $\Gamma = \{x_i : F_i\}_1^m$ and $\mu_i \succ F_i$ for $i \in [1..m]$.*

Definition 4.5 *A store $\mathcal{S}$ is well typed with respect to a store typing Σ, written $\models \mathcal{S} : \Sigma$, if and only if $dom(\mathcal{S}) = dom(\Sigma)$ and $\Sigma; \varnothing \vdash \mathcal{S}(l) : \Sigma(l)$ for every $l \in dom(\mathcal{S})$.*

Definition 4.6 *An environment $\mathcal{E}$ is well typed with respect to a store typing Σ and a type environment scheme Δ, written $\Sigma \models \mathcal{E} : \Delta$, if and only if*

$$dom(\mathcal{E}) = dom(\Delta) \quad and \quad \Sigma; \varnothing \vdash \mathcal{E}(x) : A$$

for every $x \in dom(\Delta)$ and every A where $\Delta(x) \succ A$.

Typing judgment

$$\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$$

means that an expression e, under store typing Σ and type scheme environments $\Delta_0 \cdots \Delta_n$, has type A at stage n. $\Delta_0 \cdots \Delta_n$ is a sequence of type scheme environments $\Delta_0, \ldots, \Delta_n$. Δ_n is the current type scheme environment. Subscripts $0, \ldots, n$ are stage numbers.

`(TBOX)`, `(TOPEN)` and `(TLIFT)` restrict code template type $\square(\Gamma \triangleright A)$ to be conditioned by monomorphic type environment Γ. This restriction, which is analogous to the rank-1 polymorphism, allows us to avoid the impossible *principal typing* in the Hindley/Milner-style type inference [25].

`(TBOX)`, `(TOPEN)` and `(TLIFT)` introduce type environment variables. Consider the `(TBOX)` case. Suppose that

$$\Delta_0 \cdots \Delta_n\{x_i : A_i\}_1^m \vdash e : A.$$

Then,

$$\Delta_0 \cdots \Delta_n\{x_i : A_i\}_1^m :: \{y_i : F_i\}_1^k \vdash e : A \quad \text{for some } \{y_i : F_i\}_1^k.$$

Hence, using a type environment variable ρ,

$$\Delta_0 \cdots \Delta_n\{x_i : A_i\}_1^m\rho \vdash e : A$$

or

$$\Delta_0 \cdots \Delta_n \vdash \texttt{box}\, e : \square(\{x_i : A_i\}_1^m\rho \triangleright A).$$

`(TOPEN)` and `(TLIFT)` also introduce type environment variables as follows.

$$\Delta_0 \cdots \Delta_n \vdash \texttt{open}\, e : \square(\rho \triangleright A) \quad \text{for some } \rho \quad \text{and}$$
$$\Delta_0 \cdots \Delta_n \vdash \texttt{lift}\, e : \square(\rho \triangleright A) \quad \text{for some } \rho.$$

In typing judgment $\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$, Δ_0 always has the form of $\{x_i : \tau_i\}_1^m$ (not $\{x_i : \tau_i\}_1^m\rho$) while Δ_i $(i > 0)$ may be $\{y_i : \mu_i\}_1^k\rho$. This is because type environment variables are introduced only into code template types (by `(TBOX)`, `(TOPEN)` and `(TLIFT)`) and type environment $\{x_i : F_i\}_1^m\rho$ in a code template type $\square(\{x_i : F_i\}_1^m\rho \triangleright A)$ can be Δ_i $(i > 0)$ in a type derivation.

`(TLETAPP)` allows the let-polymorphism in $\texttt{let}\,(x\ e_1)\ e_2$ if e_1 guarantees not to expand the store (if expansive$^n(e_1)$ is false). Note that type environment variables and field variables, as well as type variables, can be generalized in `(TLETAPP)`. If the evaluation of e_1 does not expand the store, it is safe to generalize free type variables, free type environment variables, or free field variables in the type of e_1. In `(TVAR)` and `(TUNBOX)`, such generalized type variables, type environment variables and field variables respectively instantiate to some types, type environments and fields. For example, the type environment variable ρ in $\forall\rho.\square(\{x_i : F_i\}_1^m\rho \triangleright A)$ can instantiate to some type environment $\{y_i : G_i\}_1^k$ (or $\{y_i : G_i\}_1^k\rho'$) where $\{x_i\}_1^m \cap \{y_i\}_1^k = \varnothing$. Other rules are the same as in λ_{open}^{sim} except for this polymorphic extension.

$$\texttt{(TCON)} \qquad \Sigma; \Delta_0 \cdots \Delta_n \vdash c : \iota$$

$$\texttt{(TVAR)} \qquad \frac{\Delta_n(x) \succ A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash x : A}$$

$$\texttt{(TABS)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n + x : A \vdash e : B}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \lambda x.e : A \to B}$$

$$\texttt{(TAPP)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 : A \to B \qquad \Sigma; \Delta_0 \cdots \Delta_n \vdash e_2 : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 e_2 : B}$$

$$\texttt{(TBOX)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \Gamma \vdash e : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{box}\, e : \Box(\Gamma \triangleright A)}$$

$$\texttt{(TUNBOX)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e : \Box(\Gamma \triangleright A) \quad \Delta_{n+k} \succ \Gamma \quad k > 0}{\Sigma; \Delta_0 \cdots \Delta_n \cdots \Delta_{n+k} \vdash \texttt{unbox}_k\, e : A}$$

$$\texttt{(TEVAL)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e : \Box(\varnothing \triangleright A)}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{unbox}_0\, e : A}$$

$$\texttt{(TOPEN)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e : \Box(\varnothing \triangleright A)}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{open}\ e : \Box(\Gamma \triangleright A)}$$

$$\texttt{(TLIFT)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{lift}\, e : \Box(\Gamma \triangleright A)}$$

$$\texttt{(TGENSYM)} \qquad \frac{\begin{array}{c}\Sigma; \Delta_0 \cdots \Delta_n + w : A \vdash [x^n \overset{n}{\mapsto} w]\, e : B \\ w \text{ not in } (\Sigma, \Delta_0 \cdots \Delta_n, \lambda^* x.e')\end{array}}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \lambda^* x.e : A \to B}$$

$$\texttt{(TREF)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{ref}\, e : A\, \texttt{ref}}$$

$$\texttt{(TDEREF)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A\, \texttt{ref}}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{!}\, e : A}$$

$$\texttt{(TASSIGN)} \qquad \frac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 : A\, \texttt{ref} \qquad \Sigma; \Delta_0 \cdots \Delta_n \vdash e_2 : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 \texttt{:=} e_2 : A}$$

$$\texttt{(TLOC)} \qquad \frac{\Sigma(l) = A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash l : A\, \texttt{ref}}$$

$$\texttt{(TCLOS)} \qquad \frac{\Sigma \models \mathcal{E} : \Delta \quad \Sigma; \Delta + x : A \vdash e : B}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{clos}(x, e, \mathcal{E}) : A \to B}$$

$$\texttt{(TLETIMP)} \qquad \frac{\begin{array}{c}\text{expansive}^n(e_1) \\ \Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 : A \quad \Sigma; \Delta_0 \cdots \Delta_n + x : A \vdash e_2 : B\end{array}}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{let}\ (x\ e_1)\ e_2 : B}$$

$$\texttt{(TLETAPP)} \qquad \frac{\begin{array}{c}\neg\, \text{expansive}^n(e_1) \\ \Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 : A \\ \Sigma; \Delta_0 \cdots \Delta_n + x : GEN_A(\Sigma, \Delta_0 \cdots \Delta_n) \vdash e_2 : B\end{array}}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{let}\ (x\ e_1)\ e_2 : B}$$

$$GEN_A(\Sigma, \Delta_0 \cdots \Delta_n) = \forall \xi_1 \ldots \xi_m.A \text{ such that}$$
$$\{\xi_1, \ldots, \xi_m\} = \mathrm{FV}(A) \setminus (\mathrm{FV}(\Sigma) \cup \textstyle\bigcup_{i=0}^{n} \mathrm{FV}(\Delta_i))$$

Figure 7: Polymorphic type system for λ^{poly}_{open}

4.5 Soundness of Type System

`(EEVAL)` at stage 0 converts $\texttt{box}\, v^1$ into v^1 at stage 0; it demotes the n-th staged value into the $(n-1)$-th staged expression. The following demotion lemma shows that demotion preserves types.

Lemma 4.3 *(Demotion) If* $\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash v : A$ *where* $n > 0$ *, then*

$$\Sigma; \Delta_1 \cdots \Delta_n \vdash v : A.$$

PROOF By induction on the type derivation of $\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash v : A$. For a given type derivation, we proceed by cases on the finally used type rule. Predicate expansive$^n(e)$'s demotion-closedness is needed in the proof. □

An expression preserves its type after evaluation in λ^{poly}_{open}. For the proof of the preservation lemma, we need the result lemma (Lemma 4.2) and the demotion lemma (Lemma 4.3).

Lemma 4.4 *(Preservation) If* $\models \mathcal{S} : \Sigma$, $\Sigma \models \mathcal{E} : \Delta_0$, *If*

$$\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A, \quad \textit{and} \quad \mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}'),$$

then,

$$\Sigma'; \varnothing\Delta_1 \cdots \Delta_n \vdash r : A \quad \textit{and} \quad \models \mathcal{S}' : \Sigma' \textit{ for some } \Sigma' \supseteq \Sigma.$$

PROOF By induction on the type derivation of $\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$ and evaluation $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}')$. For a given type derivation, we proceed by cases on the finally used type rule. □

Finally, the type system of λ^{poly}_{open} is sound. If a closed expression is well typed, then it preserves the type after evaluation. Hence, the evaluation result can not be `err`. Note that `err` is not typable.

Theorem 4.1 *(Soundness) If* $\varnothing; \varnothing \vdash e : A$ *and* $\varnothing, \varnothing, \mathcal{V} \vdash e \xrightarrow{0} (r, \mathcal{S}, \mathcal{V}')$, *then* $\Sigma; \varnothing \vdash r : A$ *and* $\models \mathcal{S} : \Sigma$.

PROOF The empty store is well typed with respect to the empty store typing: $\models \varnothing : \varnothing$. The empty environment is also well typed with respect to the empty type scheme environment and the empty store typing: $\varnothing \models \varnothing : \varnothing$. From the assumptions and the preservation lemma, it follows that $\Sigma; \varnothing \vdash r : A$ where $\models \mathcal{S} : \Sigma$. □

5 Type Inference Algorithm

This section presents a sound and complete type inference algorithm that finds the principal types for λ^{poly}_{open}. Figure 10 shows the type inference algorithm for λ^{poly}_{open}. Algorithm $\texttt{infer}(\Delta_0 \cdots \Delta_n, e, A, \mathcal{Q})$ takes as input a sequence of type scheme environments $\Delta_0 \cdots \Delta_n$, an expression e, a type A and a set $\mathcal{Q}$ of fresh type, type environment or field variables to be used in the `infer` algorithm. The `infer` algorithm finds the most general substitution R satisfying

$$\varnothing; (R\Delta_0) \cdots (R\Delta_n) \vdash e : RA.$$

Figure 11 shows the extension of type environment or field addition operation in record types. In $\varnothing; \Delta_0 \cdots \Delta_n \vdash e : A$, Δ_i is a record type which may contain ρ variable for $i > 0$. In `(IABS)`, `(IGENSYM)` and `(ILET)`, we need to extend Δ_n with $x : \tau$, meaning that $\Delta_n + x : \tau$ comes to have $x : \tau$ whether Δ_n has some $x : \theta$ or not. To support such field addition in record types, we adopt Remy's record types [20], and we allow a field variable to be $\perp$ in records.

$\texttt{unify}(E, \mathcal{Q}) =$
  We use Rémy's unification algorithm [20, 19].
  Only difference is:
    for equation $\Box(\Gamma_1 \triangleright A_1) = \Box(\Gamma_2 \triangleright A_2)$ during unification,
      we convert it into $\Gamma_1 = \Gamma_2$ and $A_1 = A_2$.
    and for equation A_1 `ref` $= A_2$ `ref` in E,
      we convert it into $A_1 = A_2$.

Figure 8: Unification algorithm

(* $\texttt{inst}(\Delta, \mathcal{Q})$ finds a type environment Γ that satisfies $\Delta \succ \Gamma$
    using fresh variables in $\mathcal{Q}$. *)

$\texttt{inst}(\Delta, \mathcal{Q}) =$
  make distinct all bound variables in Δ by renaming the bound variables
  if $\Delta = \{x_i : \mu_i\}_1^m \rho$ then
    let S be a substitution such that $dom(S) = \bigoplus_{i=1}^m \mathrm{BV}(\mu_i)$ and
      $range(S) \subseteq \mathcal{Q}$ in
    let $\mu_i \succ F_i$ by S for $i \in [1..m]$ in
      $\{x_i : F_i\}_1^m \rho$
  if $\Delta = \{x_i : \mu_i\}_1^m$ then
    let S be a substitution such that $dom(S) = \bigoplus_{i=1}^m \mathrm{BV}(\mu_i)$ and
      $range(S) \subseteq \mathcal{Q}$ in
    let $\mu_i \succ F_i$ by S for $i \in [1..m]$ in
    $\{x_i : F_i\}_1^m$

Figure 9: Instantiation of type scheme environments

Meanwhile, Rémy uses the different notion for the field in record types such as $x : \texttt{pre}(A)$ (not $x : A$) or $x : \mathrm{abs}$ (not $x : \bot$).

Our unification algorithm (Figure 8) is basically the same as Rémy's [20, 19] except for the code and reference types. Remy's unification algorithm takes as input a set of equations for types, fields or records(type environments). The second input $\mathcal{Q}$ is the set of fresh variables that will be used during the unification. The algorithm returns the most general unifier [20, 19].

`(IBOX)`, `(IOPEN)` and `(ILIFT)` introduce type environment variables into code template types. As mentioned in Section 4, in a sequence of type scheme environments $\Delta_0 \cdots \Delta_n$, Δ_n ($n > 0$) may be $\{x_i : F_i\}_1^m \rho$. Hence, to infer the type of x, `(IVAR)` checks $\Delta_n \succ \{x : A\}\rho$ (not $\Delta_n(x) \succ A$) because it may be that $\Delta_n = \{x_i : F_i\}_1^m \rho'$ and $x \notin \{x_i\}_1^m$. `(ILET)` generalizes free type environment variables and free field variables, as well as free type variables. Such generalized type, field or type environment variables instantiate to concrete types, fields or type environments in `(IVAR)` and `(IUNBOX)`. `(IVAR)` and `(IUNBOX)` need the instantiation of type scheme environments, whose algorithm is presented in Figure 9. `(IBOX)` and `(IUNBOX)` relate two different staged type scheme environments with each other by making the sequence of type scheme environments longer or shorter. `(IGENSYM)` executes the explicit name change before type inference. `(ICON)`, `(IABS)`, `(IAPP)`, `(IREF)`, `(IDEREF)` and `(IASSIGN)` are straightforward except for multi-staged setting.

The `infer` algorithm is a sound and complete type inference algorithm for λ^{poly}_{open}. The completeness lemma (Lemma 5.2) establishes that the `infer` algorithm finds the principal types of λ^{poly}_{open}.

Lemma 5.1 *(Soundness) Suppose that*

$$\mathcal{Q} \cap (\mathrm{FV}(A) \cup \bigcup_{i=0}^{n} \mathrm{FV}(\Delta_i)) = \varnothing.$$

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, c, A, \mathcal{Q}) =$ (ICON)
  $\mathtt{unify}(\{A = \iota\}, \mathcal{Q})$

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, x, A, \{\rho\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2) =$ (IVAR)
  let $\Gamma_n = \mathtt{inst}(\Delta_n, \mathcal{Q}_1)$ in
    $\mathtt{unify}(\{\Gamma_n = \{x : A\}\rho\}, \mathcal{Q}_2)$

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \lambda x.e, A, \{\alpha, \beta\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \mathcal{Q}_3) =$ (IABS)
  let $(S_1, \Delta_1' \cdots \Delta_n') = \mathtt{add}(\Delta_0 \cdots \Delta_n, \{x : \alpha\}, \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{infer}(\Delta_0' \cdots \Delta_n', e, S_1\beta, \mathcal{Q}_2)$ in
  let $S_3 = \mathtt{unify}(\{S_2S_1A = S_2S_1\alpha \to S_2S_1\beta\}, \mathcal{Q}_3)$ in $S_3S_2S_1$

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, e_1e_2, A, \{\alpha\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2) =$ (IAPP)
  let $S_1 = \mathtt{infer}(\Delta_0 \cdots \Delta_n, e_1, \alpha \to A, \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{infer}((S_1\Delta_0) \cdots (S_1\Delta_n), e_2, S_1\alpha, \mathcal{Q}_2)$ in S_2S_1

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \mathtt{box}\, e, A, \{\rho, \alpha\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2) =$ (IBOX)
  let $S_1 = \mathtt{infer}(\Delta_0 \cdots \Delta_n\rho, e, \alpha, \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{unify}(\{S_1A = \Box(S_1\rho \triangleright S_1\alpha)\}, \mathcal{Q}_2)$ in S_2S_1

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \mathtt{unbox}_k\, e, A, \{\rho\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \mathcal{Q}_3) =$ (IUNBOX)
  let $\Gamma_n = \mathtt{inst}(\Delta_n, \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{infer}(\Delta_0 \cdots \Delta_{n-k}, e, \Box(\rho \triangleright A), \mathcal{Q}_2)$ in
  let $S_3 = \mathtt{unify}(\{S_2\Gamma_n = S_2\rho\}, \mathcal{Q}_3)$ in S_3S_2

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \mathtt{unbox}_0\, e, A, \mathcal{Q}) =$ (IEVAL)
  $\mathtt{infer}(\Delta_0 \cdots \Delta_n, e, \Box(\varnothing \triangleright A), \mathcal{Q})$

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \mathtt{open}\ e, A, \{\rho, \alpha\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2) =$ (IOPEN)
  let $S_1 = \mathtt{infer}(\Delta_0 \cdots \Delta_n, e, \Box(\varnothing \triangleright \alpha), \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{unify}(\{S_1A = \Box(\rho \triangleright S_1\alpha)\}, \mathcal{Q}_2)$ in S_2S_1

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \mathtt{lift}\, e, A, \{\rho, \alpha\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2) =$ (ILIFT)
  let $S_1 = \mathtt{infer}(\Delta_0 \cdots \Delta_n, e, \alpha, \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{unify}(\{S_1A = \Box(\rho \triangleright S_1\alpha)\}, \mathcal{Q}_2)$ in S_2S_1

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \lambda^* x.e, A, \{\alpha, \beta\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \mathcal{Q}_3) =$ (IGENSYM)
  w is a fresh internal program variable
  let $(S_1, \Delta_0' \cdots \Delta_n') = \mathtt{add}(\Delta_0 \cdots \Delta_n, \{w : \alpha\}, \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{infer}(\Delta_0' \cdots \Delta_n', [x^n \overset{n}{\mapsto} w]\, e, S_1\beta, \mathcal{Q}_2)$ in
  let $S_3 = \mathtt{unify}(\{S_2S_1A = S_2S_1\alpha \to S_2S_1\beta\}, \mathcal{Q}_3)$ in $S_3S_2S_1$

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \mathtt{ref}\, e, A, \{\alpha\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2) =$ (IREF)
  let $S_1 = \mathtt{infer}(\Delta_0 \cdots \Delta_n, e, \alpha, \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{unify}(\{S_1A = (S_1\alpha)\, \mathtt{ref}\}, \mathcal{Q}_2)$ in S_2S_1

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, !\, e, A, \mathcal{Q}) =$ (IDEREF)
  $\mathtt{infer}(\Delta_0 \cdots \Delta_n, e, A\, \mathtt{ref}, \mathcal{Q})$

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, e_1 \mathtt{:=} e_2, A, \mathcal{Q}_1 \oplus \mathcal{Q}_2) =$ (IASSIGN)
  let $S_1 = \mathtt{infer}(\Delta_0 \cdots \Delta_n, e_1, A\, \mathtt{ref}, \mathcal{Q}_1)$ in
  let $S_2 = \mathtt{infer}((S_1\Delta_0) \cdots (S_1\Delta_n), e_2, S_1A, \mathcal{Q}_2)$ in S_2S_1

$\mathtt{infer}(\Delta_0 \cdots \Delta_n, \mathtt{let}\ (x\ e_1)\ e_2, A, \{\alpha\} \oplus \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \mathcal{Q}_3) =$ (ILET)
  if expansive$^n(e_1)$
  then
    let $S_1 = \mathtt{infer}(\Delta_0 \cdots \Delta_n, e_1, \alpha, \mathcal{Q}_1)$ in
    let $(S_2, \Delta_0' \cdots \Delta_n') = \mathtt{add}((S_1\Delta_0) \cdots (S_1\Delta_n), \{x : S_1\alpha\}, \mathcal{Q}_2)$ in
    let $S_3 = \mathtt{infer}(\Delta_0' \cdots \Delta_n', e_2, S_2S_1A, \mathcal{Q}_3)$ in $S_3S_2S_1$
  else
    let $S_1 = \mathtt{infer}(\Delta_0 \cdots \Delta_n, e_1, \alpha, \mathcal{Q}_1)$ in
    let $\{\xi_1, \ldots, \xi_m\} = \mathrm{FV}(S_1\alpha) \setminus \bigcup_{i=0}^{n} \mathrm{FV}(S_1\Delta_i)$
    let $(S_2, \Delta_0' \cdots \Delta_n') =$
        $\mathtt{add}((S_1\Delta_0) \cdots (S_1\Delta_n), \{x : \forall \xi_1 \ldots \xi_m.S_1\alpha\}, \mathcal{Q}_2)$ in
    let $S_3 = \mathtt{infer}(\Delta_0' \cdots \Delta_n', e_2, S_2S_1A, \mathcal{Q}_3)$ in $S_3S_2S_1$

Figure 10: Type inference algorithm for λ^{poly}_{open}

$$
\begin{array}{l}
\mathtt{add}(\Delta_0 \cdots \Delta_n, \{x:\tau\}, \{\theta, \rho'\}) = \\
\quad \text{if } \Delta_n = \{x:\mu\} :: \Delta' \text{ then} \\
\quad\quad (\varnothing,\ \Delta_0 \cdots \Delta_{n-1}\ (\{x:\tau\} :: \Delta')\ ) \\
\quad \text{else if } \Delta_n = \{x_i:\mu_i\}_1^k \text{ and } x \notin \{x_i\}_1^k \text{ then} \\
\quad\quad (\varnothing,\ \Delta_0 \cdots \Delta_{n-1}\ (\{x:\tau\} :: \{x_i:\mu_i\}_1^k)\ ) \\
\quad \text{else} \\
\quad\quad \text{let } \Delta_n = \{x_i:\mu_i\}_1^k :: \rho \text{ and } x \notin \{x_i\}_1^k \text{ in} \\
\quad\quad \text{let } S = \{\rho : \{x:\theta\} :: \rho'\} \text{ in} \\
\quad\quad\quad (S,\ (S\Delta_0) \cdots (S\Delta_{n-1})\ (\{x:S\tau\} :: \{x_i : S\mu_i\}_1^k :: \rho')\ )
\end{array}
$$

Figure 11: Extension of type environments

If $\mathtt{infer}(\Delta_0 \cdots \Delta_n, e, A, \mathcal{Q})$ *succeeds with* S*, then*

$$\varnothing; (S\Delta_0) \cdots (S\Delta_n) \vdash e : SA.$$

Proof By induction on each case of the `infer` algorithm. □

Lemma 5.2 *(Completeness) Suppose that*

$$\varnothing; (R\Delta_0) \cdots (R\Delta_n) \vdash e : RA$$

and $\mathcal{Q} \cap (\mathrm{FV}(A) \cup \bigcup_{i=0}^n \mathrm{FV}(\Delta_i)) = \varnothing$.
Then, $\mathtt{infer}(\Delta_0 \cdots \Delta_n, e, A, \mathcal{Q})$ *succeeds with* S *such that* $R|_{\mathcal{Q}}^- = TS|_{\mathcal{Q}}^-$ *for some* T *and* $\mathrm{RV}(S) \subseteq \mathcal{Q} \cup \mathrm{FV}(A) \cup \bigcup_{i=0}^n \mathrm{FV}(\Delta_i)$.

Proof By induction on the type derivation of

$$\varnothing; (R\Delta_0) \cdots (R\Delta_n) \vdash e : RA.$$

For a given type derivation, we proceed by cases on the finally used type rule. □

Suppose that a type variable α is not in a set of type and type environment variables $\mathcal{Q}$. Then $\mathtt{infer}(\varnothing, e, \alpha, \mathcal{Q})$ always terminates for any expression e. If $\mathtt{infer}(\varnothing, e, \alpha, \mathcal{Q})$ succeeds with S, then $\varnothing; \varnothing \vdash e : S\alpha$. Moreover, if there is a substitution R such that $\varnothing; \varnothing \vdash e : R\alpha$, then $R|_{\mathcal{Q}}^- = TS|_{\mathcal{Q}}^-$ for some T. Hence, $S\alpha$ is the principal type of e.

6 Relation to Other Multi-Staged Languages

This section compares λ_{open}^{poly} (or λ_{open}^{sim}) with other multi-staged languages such as the implicit $\lambda^{\Box}$ [7, 8], λ_{let}^{i} [3], $\lambda^{\bigcirc}$ [6] and λ_{code}^{+} [4].

6.1 Relation to the Implicit $\lambda^{\Box}$

Typeful expressions in the implicit $\lambda^{\Box}$ [7, 8] can be embedded into λ_{open}^{sim}. Figure 12 shows the type system of the implicit $\lambda^{\Box}$. The translation is straightforward because the implicit $\lambda^{\Box}$ accepts only closed code templates while λ_{open}^{sim} accepts code templates containing free variables. Expression translation from the implicit $\lambda^{\Box}$ to λ_{open}^{sim} is:

$$
\begin{array}{cc}
[\![x]\!] = x & [\![\lambda x.e]\!] = \lambda x.[\![e]\!] \\
[\![e_1 e_2]\!] = [\![e_1]\!][\![e_2]\!] & [\![\mathtt{box}\, e]\!] = \mathtt{box}\ [\![e]\!] \\
[\![\mathtt{unbox}_0\, e]\!] = \mathtt{unbox}_0 [\![e]\!] & [\![\mathtt{unbox}_k\, e]\!] = \mathtt{unbox}_k\ (\mathtt{open}\ [\![e]\!]) \\
 & \text{where } k > 0.
\end{array}
$$

Type translation from the implicit $\lambda^{\Box}$ to λ_{open}^{sim} is:

$$\begin{array}{lccl}
\text{Expressions} & e & ::= & x \mid \lambda x.e \mid e\ e \mid \mathtt{box}\ e \mid \mathtt{unbox}_k\ e \\
\text{Types} & A, B & ::= & \iota \mid A \to B \mid \Box A \\
\text{Type Environments} & \Gamma & ::= & \varnothing \mid \Gamma + x : A
\end{array}$$

$$\frac{\Gamma_n(x) = A}{\Gamma_0 \cdots \Gamma_n \vdash^i x : A} \qquad \frac{\Gamma_0 \cdots \Gamma_n + x : A \vdash^i e : B}{\Gamma_0 \cdots \Gamma_n \vdash^i \lambda x.e : A \to B}$$

$$\frac{\Gamma_0 \cdots \Gamma_n \vdash^i e_1 : A \to B \qquad \Gamma_0 \cdots \Gamma_n \vdash^i e_2 : A}{\Gamma_0 \cdots \Gamma_n \vdash^i e_1 e_2 : B}$$

$$\frac{\Gamma_0 \cdots \Gamma_n \varnothing \vdash^i e : A}{\Gamma_0 \cdots \Gamma_n \vdash^i \mathtt{box}\ e : \Box A} \qquad \frac{\Gamma_0 \cdots \Gamma_n \vdash^i e : \Box A}{\Gamma_0 \cdots \Gamma_n \cdots \Gamma_{n+k} \vdash^i \mathtt{unbox}_k\ e : A}$$

Figure 12: Type system for the implicit $\lambda^{\Box}$

$$[\![\iota]\!] = \iota \qquad [\![A \to B]\!] = [\![A]\!] \to [\![B]\!] \qquad [\![\Box A]\!] = \Box(\varnothing \triangleright [\![A]\!]).$$

Type environment translation $[\![\Gamma]\!]$ is point-wise:

$$[\![\{x_1 : A_1, \ldots, x_n : A_n\}]\!] = \{x_1 : [\![A_1]\!], \ldots, x_n : [\![A_n]\!]\}.$$

We can show that a typeful expression in the implicit $\lambda^{\Box}$ can be translated into a typeful expression in λ^{sim}_{open}.

Lemma 6.1 *If* $\Gamma_0 \cdots \Gamma_n \vdash^i e : A$ *in implicit* $\lambda^{\Box}$*, then*

$$\varnothing; [\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash [\![e]\!] : [\![A]\!]$$

in $\lambda^{\mathrm{sim}}_{\mathrm{open}}$.

Proof By induction on the type derivation of $\Gamma_0 \cdots \Gamma_n \vdash^i e : A$ in the implicit $\lambda^{\Box}$ language. For a given type derivation, we proceed by cases on the finally used type rule. □

6.2 Relation to λ^i_{let}

λ^i_{let} [3] is not embedded in λ^{poly}_{open} while its monomorphic version λ^i [3] is embedded in λ^{sim}_{open} if its cross-stage persistence operator (`%`) is removed. Figure 13 shows the type system of λ^i_{let}. We omit the type system of λ^i, which is just a simple type version of λ^i_{let} and does not have expression $\mathtt{let}\ (x\ e_1)\ e_2$.

Expression translation from λ^i without `%` to λ^{sim}_{open} is straightforward. Because λ^i preserves the alpha-equivalence, we assume without loss of generality that every bound variables are distinct in λ^i before execution.

$$\begin{array}{rlrl}
[\![x]\!] &= x & [\![\lambda x.e]\!] &= \lambda^* x.[\![e]\!] \\
[\![e_1 e_2]\!] &= [\![e_1]\!][\![e_2]\!] & [\![\langle e \rangle]\!] &= \mathtt{box}\ [\![e]\!] \\
[\![\tilde{}e]\!] &= \mathtt{unbox}[\![e]\!] & [\![\mathtt{run}\ e]\!] &= \mathtt{unbox}_0[\![e]\!] \\
[\![\mathtt{open}\ e]\!] &= \mathtt{open}\ [\![e]\!] & [\![\mathtt{close}\ e]\!] &= [\![e]\!]
\end{array}$$

Note that because runtime renaming that happens implicitly in λ^i is simulated by our hygienic abstraction $\lambda^* x.e$, the translated version in λ^{sim}_{open} has the same semantics as the original one in λ^i.

Definition 6.1 *(Collecting Type Environments) Let* $\{\Gamma_i \vdash^{L_i} e_i : A_i\}_1^m$ *be a set of typing judgments occurring in* λ^i*'s type derivation tree of* $\Gamma \vdash^L e : A$

$$\mathrm{CTE}(\Gamma \vdash^L e : A) = \textstyle\bigcup_{i=1}^m \Gamma_i.$$

$$
\begin{array}{lrcl}
\text{Expressions} & e & ::= & x \mid \lambda x.e \mid e_1 e_2 \\
& & \mid & \langle e \rangle \mid \tilde{}\, e \mid \% \, e \mid \mathtt{run}\, e \\
& & \mid & \mathtt{open}\, e \mid \mathtt{close}\, e \\
& & \mid & \mathtt{let}\, (x\ e_1)\ e_2 \\
\text{Types} & A, B & ::= & \iota \mid A \to B \mid \langle A \rangle^{\alpha} \mid \langle A \rangle \\
\text{Type Schemes} & \tau & ::= & A \mid \forall \kappa.\tau \\
\text{Type Environments} & \Gamma & ::= & \varnothing \mid \Gamma + x : A^L \\
\text{Type Scheme Environments} & \Delta & ::= & \varnothing \mid \Delta + x : \tau^L
\end{array}
$$

$$\frac{\Delta(x) = \tau^L \quad \tau \succ A}{\Delta \vdash^L x : A} \qquad \frac{\Delta + x : A^L \vdash^L e : B}{\Delta \vdash^L \lambda x.e : A \to B}$$

$$\frac{\Delta \vdash^L e_1 : A \to B \quad \Delta \vdash^L e_2 : A}{\Delta \vdash^L e_1 e_2 : B}$$

$$\frac{\Delta \vdash^{L\alpha} e : A}{\Delta \vdash^L \langle e \rangle : \langle A \rangle^{\alpha}} \qquad \frac{\Delta \vdash^L e : \langle A \rangle^{\alpha}}{\Delta \vdash^{L\alpha} \tilde{}\, e : A}$$

$$\frac{\Delta \vdash^L e : \langle A \rangle}{\Delta \vdash^L \mathtt{run}\, e : A} \qquad \frac{\Delta \vdash^L e : A}{\Delta \vdash^{L\alpha} \% \, e : A}$$

$$\frac{\Delta \vdash^L e : \langle A \rangle}{\Delta \vdash^L \mathtt{open}\, e : \langle A \rangle^{\alpha}} \qquad \frac{\Delta \vdash^L e : \langle A \rangle^{\alpha} \quad \alpha \notin \mathrm{FV}(\Delta, L, A)}{\Delta \vdash^L \mathtt{close}\, e : \langle A \rangle}$$

$$\frac{\begin{array}{l}\Delta \vdash^L e_1 : A \\ \Delta + x : \forall \kappa_1 \ldots \kappa_m . A^L \vdash^L e_2 : B \quad \{\kappa_i\}_1^m = \mathrm{FV}(A) \setminus \mathrm{FV}(\Delta, L)\end{array}}{\Delta \vdash^L \mathtt{let}\, (x\ e_1)\ e_2 : B}$$

Figure 13: Type system for λ^i_{let}

As we assume that every bound variables are distinct, if $x : A_1^{L_1}$ and $x : A_2^{L_2}$ are in $\mathrm{CTE}(\Gamma \vdash^L e : A)$, then $A_1^{L_1} = A_2^{L_2}$. In other words, $\mathrm{CTE}(\Gamma \vdash^L e : A)$ is always a function. Let Φ be a function satisfying $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^L e : A)$. Then, type translation from λ^i without `%` to λ^{sim}_{open} is:

$$
\begin{array}{rcl}
\mathtt{Tr}(\Phi, \iota, L) & = & \iota \\
\mathtt{Tr}(\Phi, A \to B, L) & = & \mathtt{Tr}(\Phi, A, L) \to \mathtt{Tr}(\Phi, B, L) \\
\mathtt{Tr}(\Phi, \langle A \rangle^{\alpha}, L) & = & \Box([\![\Phi]\!]^{L\alpha}_{|L\alpha|} \triangleright \mathtt{Tr}(\Phi, A, L\alpha)) \\
\mathtt{Tr}(\Phi, \langle A \rangle, L) & = & \Box(\varnothing \triangleright \mathtt{Tr}(\Phi, A, L')) \\
 & & \textit{where } L' \textit{ is any stage,}
\end{array}
$$

and environment translation is:

$$
\begin{array}{rl}
[\![\Phi]\!]^L_i = \{x : \mathtt{Tr}(\Phi, A, L') \mid & L' \textit{ is a prefix of } L,\ |L'| = i, \\
& x : A^{L'} \in \Phi\}
\end{array}
$$

Lemma 6.2 *If* $\Gamma \vdash^L e : A$ *in* λ^i *without* `%`*, then*

$$[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![e]\!] : \mathtt{Tr}(\Phi, A, L) \quad \textit{in } \lambda^{\mathrm{poly}}_{\mathrm{open}},$$

for any function $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^L e : A)$.

Proof By induction on the type derivation of $\Gamma \vdash^L e : A$ in λ^i without the cross-stage persistence. For a given type derivation, we proceed by cases on the finally used type rule. □

Although translation from λ^i_{let} to λ^{poly}_{open} seems straightforward, λ^i_{let} is not conservatively embedded in λ^{poly}_{open}. The code template type of $\langle e \rangle$ is not always translated into some modal type of `box` e. For example, code template

$$\langle\ \text{let } x = \lambda y.y \text{ in } \tilde{}\langle xx \rangle\ \rangle$$

is admissible to λ^i_{let}'s type system as follows.

$$\dfrac{\dfrac{\dfrac{y : B^\alpha \vdash^\alpha y : B}{\varnothing \vdash^\alpha \lambda y.y : B \to B} \quad \dfrac{\dfrac{x : \forall\kappa.\kappa \to \kappa^\alpha \vdash^\alpha xx : A \to A}{x : \forall\kappa.\kappa \to \kappa^\alpha \vdash^\varepsilon \langle xx \rangle : \langle A \to A \rangle^\alpha}}{x : \forall\kappa.\kappa \to \kappa^\alpha \vdash^\alpha \tilde{}\langle xx \rangle : A \to A}}{\varnothing \vdash^\alpha \text{let } x = \lambda y.y \text{ in } \tilde{}\langle xx \rangle : A \to A}}{\varnothing \vdash^\varepsilon \langle \text{ let } x = \lambda y.y \text{ in } \tilde{}\langle xx \rangle \rangle : \langle A \to A \rangle^\alpha}$$

In the above type derivation tree, code template $\langle xx \rangle$ is typable in λ^i_{let}. However, code template `box`(xx) is not typable in λ^{poly}_{open} because λ^{poly}_{open} allows only monomorphic type environment inside the code template types (i.e., $\Box(\Gamma \triangleright A)$ not $\Box(\Delta \triangleright A)$). Note that xx is not typable under any monomorphic type environment.

On the other hand, free variables of $\langle e \rangle$ should be lexically bound at the same stage level, which makes it difficult to support imperative operations for open code templates. For example, suppose that open code template $\langle x \rangle$ is stored at a stage L under some Δ where $x : A^{L\alpha} \in \Delta$. The stored open code template may be later dereferenced at a stage L under some Δ' ($x : A^{L\alpha} \notin \Delta'$). This situation which is typable in λ^{poly}_{open} is not so in λ^i_{let}.

6.3 Relation to $\lambda^{\bigcirc}$

$\lambda^{\bigcirc}$ [6] is embedded in λ^i [3], and does not have the cross-stage persistence operator (`%`). Because λ^i without `%` is embedded in λ^{sim}_{open} (Section 6.2), $\lambda^{\bigcirc}$ is thus embedded in λ^{sim}_{open}.

6.4 Relation to λ^+_{code}

Unlike ours, λ^+_{code} [4]'s polymorphic generalization is not allowed inside code templates, and no free named variables can occur inside code templates.

λ^+_{code}'s use of deBruijn indices for program variables conflicts with imperative multi-staged programming practice. When an open code template escapes from its lexical scope by some imperative operations, its free variables' deBruijn indices can denote different variables from those in the original program with named variables. For example, consider the following code that is admissible to λ^+_{code}'s type system:

```
let val a = ref ‘1
  val f = ‘(fn x -> fn y -> ,(a := ‘(x + y); ‘2))
  val g = ‘(fn y -> fn z -> ,(!a))
in ...  end
```

When the stored open code `‘(x+y)` is plugged inside `‘(fn y -> fn z -> ,(!a))`, the programmer intends that the `x` remains free and `y` is bound to y. If we replace every variable by its deBruijn index, the resulting program becomes a completely different one. The `x` and `y` in `’(x+y)` will have deBruijn indices 2 and 1, respectively. Thus in `‘(fn y -> fn z -> ,(!a))`, the `x` is bound with y and the `y` with z.

7 Conclusion

We have presented a polymorphic type system and its principal type inference algorithm for an imperative multi-staged language that combines ML and Lisp's staging constructs. Lisp has the longest history of staged programming in its "quasi-quote macro" system. This macro system supports open code templates, imperative operations with code templates, intensional variable-capturing substitution (unhygienic macros, at the sacrifice of the alpha-equivalence) as well as capture-avoiding substitution (hygienic macros) of free variables in open code templates, and lifting values into code templates. Our type system supports all these features.

Type-checked programs preserve the alpha-equivalence only at stage 0. At stage level 1 or more (i.e., during macro definitions and expansions) type-checked programs can violate the alpha-equivalence because of unhygienic macros. This violation, which may be unacceptable in a purely functional language, is frequently practiced in Lisp's macro programming, hence, we think, is worthwhile to be supported by our type system.

For programs that are accepted by existing multi-staged type systems such as $\lambda^{\Box}$ [7, 8], $\lambda^{\bigcirc}$ [6] and λ^{i} without the cross-stage persistence [3], there exist their semantics-preserving, translated versions that are accepted by our type system.

We believe our type system is only one supporting open code, unrestricted operations on references, both hygienic and unhygienic macros, and a type inference algorithm. Our type system conservatively extends ML's let-polymorphism for imperative multi-staged computation that handles open code templates as first-class objects. ML's rank-0 polymorphism is also orthogonally combined with a record polymorphism that aims to allow a single open code template in multiple environments.

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A Appendix

Lemma 4.3 (Demotion)
PROOF By induction on the type derivation of

$$\Sigma; \varnothing \Delta_1 \cdots \Delta_n \vdash v : A.$$

For a given type derivation, we proceed by cases on the finally used type rule.

Case (TCON): Let $v = c$.

(1) $\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash c : A$ by assumption

(2) $A = \iota$ by (1)

(3) $\Sigma; \Delta_1 \cdots \Delta_n \vdash c : A$ by (TCON)

Case (TVAR): Let $v = x$.

(1) $\dfrac{\Delta_n(x) \succ A}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash x : A}$ by assumption

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash x : A$ by the premise of (1) and (TVAR)

Case (TABS): Let $v = \lambda x.e$.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_n + x : A_1 \vdash e : A_2}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \lambda x.e : A_1 \to A_2}$ by assumption

(2) $\lambda x.e$ is in V^n where $n > 0$ by assumption

(3) e is in V^n where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n + x : A_1 \vdash e : A_2$ by (3), the premise of (1) and ind. hyp.

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \lambda x.e : A_1 \to A_2$ by (4) and (TABS)

Case (TAPP): Let $v = e_1 e_2$.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e_1 : B \to A \quad \Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e_2 : B}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e_1 e_2 : A}$ by assumption

(2) $e_1 e_2$ is in V^n where $n > 0$ by assumption

(3) e_1 and e_2 are in V^n where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e_1 : B \to A$ by (3), the first premise of (1) and ind. hyp.

(5) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e_2 : B$ by (3), the second premise of (1) and ind. hyp.

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash e_1 e_2 : A$ by (4), (5) and (TAPP)

Case (TBOX): Let $v = \mathtt{box}\, e$.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \Gamma \vdash e : A}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \mathtt{box}\, e : \Box(\Gamma \triangleright A)}$ by assumption

(2) $\mathtt{box}\, e$ is in V^n where $n > 0$ by assumption

(3) e is in V^{n+1} where $n > 0$ by (2) and def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n \Gamma \vdash e : A$ by (3), the premise of (1) and ind. hyp.

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \mathtt{box}\, e : \Box(\Gamma \triangleright A)$ by (4) and (TBOX)

Case (TUNBOX): Let $v = \mathtt{unbox}_k\, e$.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_{n-k} \vdash e : \Box(\Gamma \triangleright A) \quad \Delta_n \succ \Gamma}{\Sigma; \varnothing\Delta_1 \cdots \Delta_{n-k} \cdots \Delta_n \vdash \mathtt{unbox}_k\, e : A}$ by assumption

(2) $\mathtt{unbox}_k\, e$ is in V^n where $n > 0$ by assumption

(3) e is in V^{n-k} where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_{n-k} \vdash e : \Box(\Gamma \triangleright A)$ by (3), the first premise of (1) and ind. hyp.

Thus, $\Sigma; \Delta_1 \cdots \Delta_{n-k} \cdots \Delta_n \vdash \mathtt{unbox}_k\, e : A$ by (4), the second premise of (1) and (TUNBOX)

Case (TLIFT): Let $v = \mathtt{lift}\, e$.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e : A}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \texttt{lift}\ e : \Box(\Gamma \triangleright A)}$ by assumption

(2) $\texttt{lift}\ e$ is in V^n where $n > 0$ by assumption

(3) e is in V^n where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e : A$ by (3), the premise of (1) and ind. hyp.

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \texttt{lift}\ e : \Box(\Gamma \triangleright A)$ by (4) and (TLIFT)

Case (TGENSYM): Let $v = \lambda^* x.e$.

(1) $\dfrac{\begin{array}{c}\Sigma; \varnothing\Delta_1 \cdots \Delta_n + w : A_1 \vdash [x^n \overset{n}{\mapsto} w]\, e : A_2 \\ w \text{ not in } (\Sigma, \varnothing\Delta_1 \cdots \Delta_n, \lambda^* x.e)\end{array}}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \lambda^* x.e : A_1 \to A_2}$ by assumption

(2) $\lambda^* x.e$ is in V^n where $n > 0$ by assumption

(3) e is in V^n where $n > 0$ by (2) and the def. of V^n

(4) $[x^n \overset{n}{\mapsto} w]\, e$ is in V^n where $n > 0$ by (3)

(5) $\Sigma; \Delta_1 \cdots \Delta_n + w : A_1 \vdash [x^n \overset{n}{\mapsto} w]\, e : A_2$ by (3), the first premise of (1) and ind. hyp.

(6) $\Sigma; \Delta_1 \cdots \Delta_n + w : A_1 \vdash [x^{n-1} \overset{n-1}{\mapsto} w]\, e : A_2$ by (5)

(7) w is not in $(\Sigma, \Delta_1 \cdots \Delta_n, \lambda^* x.e)$ by the second premise of (1)

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \lambda^* x.e : A_1 \to A_2$ by (6), (7) and (TGENSYM)

Case (TREF): Let $v = \texttt{ref}\ e$.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e : A}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \texttt{ref}\ e : A\ \texttt{ref}}$ by assumption

(2) $\texttt{ref}\ e$ is in V^n where $n > 0$ by assumption

(3) e is in V^n where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e : A$ by (3), the premise of (1) and ind. hyp.

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \texttt{ref}\ e : A\ \texttt{ref}$ by (4) and (TREF)

Case (TDEREF): Let $v = \texttt{!}\ e$.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e : A\ \texttt{ref}}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \texttt{!}\ e : A}$ by assumption

(2) $\texttt{!}\ e$ is in V^n where $n > 0$ by assumption

(3) e is in V^n where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e : A\ \texttt{ref}$ by (3), the premise of (1) and ind. hyp.

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \texttt{!}\ e : A$ by (4) and (TDEREF)

Case (TASSIGN): Let $v = e_1 \texttt{:=} e_2$.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e_1 : A\ \texttt{ref} \quad \Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e_2 : A}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e_1 \texttt{:=} e_2 : A}$ by assumption

(2) $e_1 \texttt{:=} e_2$ is in V^n where $n > 0$ by assumption

(3) e_1 and e_2 are in V^n where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e_1 : A\ \texttt{ref}$ by (3), the first premise of (1) and ind. hyp.

(5) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e_2 : A$ by (3), the second premise of (1) and ind. hyp.

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash e_1 \texttt{:=} e_2 : A$ by (4), (5) and `(TASSIGN)`

Case `(TLOC)`: Let $v = l$.

(1) $\dfrac{\Sigma(l) = A}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash l : A\ \texttt{ref}}$ by assumption

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash l : A\ \texttt{ref}$ by the premise of (1) and `(TLOC)`

Case `(TCLOS)`: Let $v = \texttt{clos}(x, e, \mathcal{E})$.

(1) $\dfrac{\Sigma \models \mathcal{E} : \Delta \quad \Sigma; \Delta + x : A_1 \vdash e : A_2}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \texttt{clos}(x, e, \mathcal{E}) : A_1 \to A_2}$ by assumption

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \texttt{clos}(x, e, \mathcal{E}) : A_1 \to A_2$ by the premises of (1) and `(TCLOS)`

Case `(TLETIMP)`: Let $v = \texttt{let}\ (x\ e_1)\ e_2$ and expansive$^n(e_1)$ is true.

(1) $\dfrac{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e_1 : B \quad \Sigma; \varnothing\Delta_1 \cdots \Delta_n + x : B \vdash e_2 : A}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \texttt{let}\ (x\ e_1)\ e_2 : A}$ by assumption

(2) $\texttt{let}\ (x\ e_1)\ e_2$ is in V^n where $n > 0$ by assumption

(3) e_1 and e_2 are in V^n where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e_1 : B$ by (3), the first premise of (1) and ind. hyp.

(5) $\Sigma; \Delta_1 \cdots \Delta_n + x : B \vdash e_2 : A$ by (3), the second premise of (1) and ind. hyp.

(6) Conservatively, we assume expansive$^{n-1}(e_1)$ is true.

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \texttt{let}\ (x\ e_1)\ e_2 : A$ by (4), (5), (6) and `(TLETIMP)`

Case `(TLETAPP)`: Let $v = \texttt{let}\ (x\ e_1)\ e_2$ and expansive$^n(e_1)$ is false.

(1) $\dfrac{\begin{array}{c}\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash e_1 : B \\ \Sigma; \varnothing\Delta_1 \cdots \Delta_n + x : \mathit{GEN}_B(\Sigma, \varnothing\Delta_1 \cdots \Delta_n) \vdash e_2 : A\end{array}}{\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash \texttt{let}\ (x\ e_1)\ e_2 : A}$ by assumption

(2) $\texttt{let}\ (x\ e_1)\ e_2$ is in V^n where $n > 0$ by assumption

(3) e_1 and e_2 are in V^n where $n > 0$ by (2) and the def. of V^n

(4) $\Sigma; \Delta_1 \cdots \Delta_n \vdash e_1 : B$ by (3), the first premise of (1) and ind. hyp.

(5) $\Sigma; \Delta_1 \cdots \Delta_n + x : \mathit{GEN}_B(\Sigma, \varnothing\Delta_1 \cdots \Delta_n) \vdash e_2 : A$ by (3), the second premise of (1) and ind. hyp.

(6) $\Sigma; \Delta_1 \cdots \Delta_n + x : \mathit{GEN}_B(\Sigma, \Delta_1 \cdots \Delta_n) \vdash e_2 : A$ by (5) and def. of *GEN*

(7) expansive$^{n-1}(e_1) = \mathit{False}$ by expansive$^n(e_1) = \mathit{False}$

Thus, $\Sigma; \Delta_1 \cdots \Delta_n \vdash \texttt{let}\ (x\ e_1)\ e_2 : A$ by (4), (6), (7) and `(TLETAPP)`

□

Lemma 4.4 (Preservation)

PROOF By induction on the type derivation of $\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$. For a given type derivation, we proceed by cases on the finally used type rule. Note that $\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$ implies $r \neq \texttt{err}$ since `err` is not typable. We shortly write "ind. hyp." for induction hypothesis, "inv." for inversion, and `(EERR)` for `err`-generating rules.

Case `(TCON)`: Let $e = c$.

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma; \Delta_0 \cdots \Delta_n \vdash c : A$ by assumption

(3) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash c \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}')$ for some r, $\mathcal{S}'$ and $\mathcal{V}'$ by assumption

(4) $r = c$, $\mathcal{S}' = \mathcal{S}$ and $\mathcal{V}' = \mathcal{V}$ by (3) and (ECON)

(5) $A = \iota$ by (2) and (TCON)

(6) $\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash c : \iota$ by (TCON)

Thus, $\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash r : A$ and $\models \mathcal{S} : \Sigma$ by (6), (5), (4) and (1)

Case (TVAR): Let $e = x$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Delta_0(x) \succ A}{\Sigma; \Delta_0 \vdash x : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash x \xrightarrow{0} (r, \mathcal{S}', \mathcal{V}')$ for some r, $\mathcal{S}'$ and $\mathcal{V}'$ by assumption

(5) $x \in dom(\mathcal{E})$ by (2) and the premise of (3)

(6) $r = \mathcal{E}(x)$, $\mathcal{S}' = \mathcal{S}$ and $\mathcal{V}' = \mathcal{V}$ by (4), (5) and (EVAR)

(7) $\Sigma; \varnothing \vdash \mathcal{E}(x) : A$ by (2), (5) and the premise of (3)

Thus, $\Sigma; \varnothing \vdash r : A$ and $\models \mathcal{S} : \Sigma$ by (7), (6) and (1)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Delta_n(x) \succ A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash x : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash x \xrightarrow{n} (r, \mathcal{S}', \mathcal{V}')$ for some r, $\mathcal{S}'$ and $\mathcal{V}'$ by assumption

(5) $r = x$, $\mathcal{S}' = \mathcal{S}$ and $\mathcal{V}' = \mathcal{V}$ by (4) and (EVAR)

(6) $\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash x : A$ by the premise of (3) and (TVAR)

Thus, $\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash r : A$ and $\models \mathcal{S} : \Sigma$ by (6), (5) and (1)

Case (TABS): Let $e = \lambda x.e'$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n + x : A_1 \vdash e : A_2}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \lambda x.e : A_1 \to A_2}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \lambda x.e' \xrightarrow{0} (r, \mathcal{S}', \mathcal{V}')$ for some r, $\mathcal{S}'$ and $\mathcal{V}'$ by assumption

(5) $r = \mathtt{clos}(x, e', \mathcal{E})$, $\mathcal{S}' = \mathcal{S}$ and $\mathcal{V}' = \mathcal{V}$ by (4) and (EABS)

(6) $\Sigma; \Delta \vdash \mathtt{clos}(x, e', \mathcal{E}) : A$ for any Δ by (2), the premise of (3) and (TCLOS)

Thus, $\Sigma; \varnothing \vdash r : A$ and $\models \mathcal{S} : \Sigma$ by (6), (5) and (1)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n + x : A_1 \vdash e' : A_2}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \lambda x.e' : A_1 \to A_2}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \lambda x.e' \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{n} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (EABS)/(EERR)

(6) $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n + x : A_1 \vdash r_2 : A_2$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) $r_1 = \lambda x.r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and (EABS)

(9) $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n \vdash \lambda x.r_2 : A_1 \to A_2$ by (6) and (TABS)

Thus, $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n \vdash r_1 : A_1 \to A_2$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (9), (8) and (7)

Case (TAPP): Let $e = e_1 e_2$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \vdash e_1 : B \to A \qquad \Sigma; \Delta_0 \vdash e_2 : B}{\Sigma; \Delta_0 \vdash e_1 e_2 : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 e_2 \xrightarrow{0} (r_3, \mathcal{S}_3, \mathcal{V}_3)$ for some r_3, $\mathcal{S}_3$ and $\mathcal{V}_3$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{0} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4) and (EAPP)/(EERR)

(6) $\Sigma_4; \varnothing \vdash r_4 : B \to A$ and

(7) $\models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$
by (1), (2), the first premise of (3), (5) and ind. hyp.

(8) r_4 is in $V^0 \oplus \{\mathtt{err}\}$ by (5) and Lemma 4.2

(9) $r_4 = \mathtt{clos}(x, e', \mathcal{E}')$ for some x, e' and $\mathcal{E}'$ by (6) and (8)

(10) $\mathcal{E}, \mathcal{S}_4, \mathcal{V}_4 \vdash e_2 \xrightarrow{0} (r_5, \mathcal{S}_5, \mathcal{V}_5)$ for some r_5, $\mathcal{S}_5$ and $\mathcal{V}_5$
by (4), (5), (6), (7), (9) and (EAPP)/(EERR)

(11) $\Sigma_4; \Delta_0 \vdash e_2 : B$ by (7), the second premise of (3) and Lemma A.1

(12) $\Sigma_4 \models \mathcal{E} : \Delta_0$ by (2), (7) and Lemma A.2

(13) $\Sigma_5; \varnothing \vdash r_5 : B$ and

(14) $\models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma_4$ by (7), (12), (11), (10) and ind. hyp.

(15) $\mathcal{E}' + x : r_5, \mathcal{S}_5, \mathcal{V}_5 \vdash e' \xrightarrow{0} (r_6, \mathcal{S}_6, \mathcal{V}_6)$ for some r_6, $\mathcal{S}_6$ and $\mathcal{V}_6$
by (4), (5), (6), (7), (9), (10), (13), (14) and (EAPP)/(EERR)

(16) $\Sigma_5; \varnothing \vdash \mathtt{clos}(x, e', \mathcal{E}') : B \to A$ by (6), (9), (14) and Lemma A.1

(17) $\Sigma_5 \models \mathcal{E}' : \Delta$ and

(18) $\Sigma_5; \Delta + x : B \vdash e' : A$ for some Δ by (16) and inv. of (TCLOS)

(19) $\Sigma_5 \models (\mathcal{E}' + x : r_5) : (\Delta + x : B)$ by (13) and (17)

(20) $\Sigma_6; \varnothing \vdash r_6 : A$ and

(21) $\models \mathcal{S}_6 : \Sigma_6$ for some $\Sigma_6 \supseteq \Sigma_5$ by (14), (19), (18), (15) and ind. hyp.

(22) $r_3 = r_6$, $\mathcal{S}_3 = \mathcal{S}_6$ and $\mathcal{V}_3 = \mathcal{V}_6$
by (4), (5), (6), (7), (9), (10), (13), (14), (15), (20), (21) and (EAPP)

Thus, $\Sigma_6; \varnothing \vdash r_3 : A$ and $\models \mathcal{S}_3 : \Sigma_6$ for some $\Sigma_6 \supseteq \Sigma$
by (20), (22), (21), (14) and (7)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 : B \to A \quad \Sigma; \Delta_0 \cdots \Delta_n \vdash e_2 : B}{\Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 e_2 : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 e_2 \xrightarrow{n} (r_3, \mathcal{S}_3, \mathcal{V}_3)$ for some r_3, $\mathcal{S}_3$ and $\mathcal{V}_3$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{n} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4) and `(EAPP)`/`(EERR)`

(6) $\Sigma_4; \varnothing \Delta_1 \cdots \Delta_n \vdash r_4 : B \to A$ and

(7) $\models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$
by (1), (2), the first premise of (3), (5) and ind. hyp.

(8) $\mathcal{E}, \mathcal{S}_4, \mathcal{V}_4 \vdash e_2 \xrightarrow{n} (r_5, \mathcal{S}_5, \mathcal{V}_5)$ for some r_5, $\mathcal{S}_5$ and $\mathcal{V}_5$
by (4), (5), (6), (7) and `(EAPP)`/`(EERR)`

(9) $\Sigma_4; \Delta_0 \cdots \Delta_n \vdash e_2 : B$ by (7), the second premise of (3) and Lemma A.1

(10) $\Sigma_4 \models \mathcal{E} : \Delta_0$ by (2), (7) and Lemma A.2

(11) $\Sigma_5; \varnothing \Delta_1 \cdots \Delta_n \vdash r_5 : B$ and

(12) $\models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma_4$ by (7), (10), (9), (8) and ind. hyp.

(13) $r_3 = r_4 r_5$, $\mathcal{S}_3 = \mathcal{S}_5$ and $\mathcal{V}_3 = \mathcal{V}_5$
by (4), (5), (6), (7), (8), (11), (12) and `(EAPP)`

(14) $\Sigma_5; \varnothing \Delta_1 \cdots \Delta_n \vdash r_4 : B \to A$ by (6), (12) and Lemma A.1

(15) $\Sigma_5; \varnothing \Delta_1 \cdots \Delta_n \vdash r_4 r_5 : A$ by (11), (14) and `(TAPP)`

Thus, $\Sigma_5; \varnothing \Delta_1 \cdots \Delta_n \vdash r_3 : A$ and $\models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma$
by (15), (13), (12) and (7)

Case `(TBOX)`: Let $e = \texttt{box}\, e'$.

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n \Gamma \vdash e' : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{box}\, e' : \Box(\Gamma \triangleright A)}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{box}\, e' \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{n+1} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and `(EBOX)`/`(EERR)`

(6) $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n \Gamma \vdash r_2 : A$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) $r_1 = \texttt{box}\, r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and `(EBOX)`

(9) $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n \vdash \texttt{box}\, r_2 : \Box(\Gamma \triangleright A)$ by (6) and `(TBOX)`

Thus, $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n \vdash r_1 : \Box(\Gamma \triangleright A)$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (9), (8) and (7)

Case (TUNBOX): Let $e = \mathtt{unbox}_k\, e'$.
If $n = k = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \vdash e' : \Box(\Gamma_0 \triangleright A) \quad \Delta_0 \succ \Gamma_0}{\Sigma; \Delta_0 \vdash \mathtt{unbox}_k\, e' : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \mathtt{unbox}_0\, e' \xrightarrow{0} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{0} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (EEVAL)/(EERR)

(6) $\Sigma_2; \varnothing \vdash r_2 : \Box(\Gamma_0 \triangleright A)$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the first premise of (3), (5) and ind. hyp.

(8) r_2 is in $V^0 \oplus \{\mathtt{err}\}$ by (5) and Lemma 4.2

(9) $r_2 = \mathtt{box}\, v_3$ for some v_3 in V^1 by (6), (8) and the def. of V^0

(10) $\mathcal{E}, \mathcal{S}_2, \mathcal{V}_2 \vdash v_3 \xrightarrow{0} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4), (5), (6), (7), (9) and (EEVAL)/(EERR)

(11) $\Sigma_2; \varnothing\Gamma_0 \vdash v_3 : A$ by (6), (9) and inv. of (TBOX)

(12) $\Sigma_2; \Gamma_0 \vdash v_3 : A$ by (11), (9) and Lemma 4.3

(13) $\Sigma \models \mathcal{E} : \Gamma_0$ by (2) and the second premise of (3)

(14) $\Sigma_2 \models \mathcal{E} : \Gamma_0$ by (7), (13) and Lemma A.2

(15) $\Sigma_4; \varnothing \vdash r_4 : A$ and

(16) $\models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma_2$ by (7), (14), (12), (10) and ind. hyp.

(17) $r_1 = r_4$, $\mathcal{S}_1 = \mathcal{S}_4$ and $\mathcal{V}_1 = \mathcal{V}_4$
by (4), (5), (6), (7), (9), (10), (15), (16) and (EEVAL)

Thus, $\Sigma_4; \varnothing \vdash r_1 : A$ and $\models \mathcal{S}_1 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$ by (17), (16) and (7)

If $n = k > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \vdash e' : \Box(\Gamma_n \triangleright A) \quad \Delta_n \succ \Gamma_n}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \mathtt{unbox}_k\, e' : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \mathtt{unbox}_k\, e' \xrightarrow{k} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{0} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (EUNBOX)/(EERR)

(6) $\Sigma_2; \varnothing \vdash r_2 : \Box(\Gamma_0 \triangleright A)$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the first premise of (3), (5) and ind. hyp.

(8) r_2 is in $V^0 \oplus \{\mathtt{err}\}$ by (5) and Lemma 4.2

(9) $r_2 = \mathtt{box}\, v_3$ for some v_3 in V^1 by (8), (6) and the def. of V^0

(10) $\Sigma_2; \varnothing\Gamma_n \vdash v_3 : A$ by (6), (9) and inv. of (TBOX)

(11) $\Sigma_2; \varnothing\Delta_n \vdash v_3 : A$ by (10) and the second premise of (3)

(12) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash v_3 : A$ by (11) and (9)

(13) $r_1 = v_3$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7), (9), (12) and (EUNBOX)

Thus, $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_1 : A$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (12), (13) and (7)

If $n > k \geq 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_{n-k} \vdash e' : \Box(\Gamma_n \triangleright A) \quad \Delta_n \succ \Gamma_n}{\Sigma; \Delta_0 \cdots \Delta_{n-k} \cdots \Delta_n \vdash \mathtt{unbox}_k\, e' : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \mathtt{unbox}_k\, e' \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{n-k} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$ by (4) and (EEVAL)/(EUNBOX)/(EERR)

(6) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_{n-k} \vdash r_2 : \Box(\Gamma_n \triangleright A)$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (1), (2), the first premise of (3), (5) and ind. hyp.

(8) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_{n-k} \cdots \Delta_n \vdash \mathtt{unbox}_k\, r_2 : A$ by (6), the second premise of (3) and (TUNBOX)

(9) $r_1 = \mathtt{unbox}_k\, r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and ind. hyp.

Thus, $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_{n-k} \cdots \Delta_n \vdash r_1 : A$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (8), (9) and (7)

Case (TOPEN): Let $e = \mathtt{open}\, e'$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \vdash e' : \Box(\varnothing \triangleright A)}{\Sigma; \Delta_0 \vdash \mathtt{open}\, e' : \Box(\Gamma \triangleright A)}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \mathtt{open}\, e' \xrightarrow{0} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{0} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$ by (4) and (EOPEN)/(EERR)

(6) $\Sigma_2; \varnothing \vdash r_2 : \Box(\varnothing \triangleright A)$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (1), (2), the premise of (3), (5) and ind. hyp.

(8) r_2 is in $V^0 \oplus \{\mathtt{err}\}$ by (5) and Lemma 4.2

(9) $r_2 = \mathtt{box}\, v$ for some $v \in V^1$ by (6), (8) and the def. of V^0

(10) $\Sigma_2; \varnothing\varnothing \vdash v : A$ by (6), (9) and inv. of (TBOX)

(11) $\Sigma_2; \varnothing\Gamma \vdash v : A$ by (10) and weakening

(12) $\Sigma_2; \varnothing \vdash \mathtt{box}\, v : \Box(\Gamma \triangleright A)$ by (11) and (TBOX)

(13) $\Sigma_2; \varnothing \vdash r_2 : \Box(\Gamma \triangleright A)$ by (12) and (9)

(14) $r_1 = r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and (EOPEN)

Thus, $\Sigma_2; \varnothing \vdash r_1 : \Box(\Gamma \triangleright A)$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (13), (14) and (7)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e' : \Box(\varnothing \triangleright A)}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{open}\, e' : \Box(\Gamma \triangleright A)}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{open}\, e' \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{n} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (EOPEN)/(EERR)

(6) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_2 : \Box(\varnothing \triangleright A)$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash \texttt{open}\, r_2 : \Box(\Gamma \triangleright A)$ by (6) and (TOPEN)

(9) $r_1 = \texttt{open}\, r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and (EOPEN)

Thus, $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_1 : \Box(\Gamma \triangleright A)$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (8), (9) and (7)

Case (TLIFT): Let $e = \texttt{lift}\, e'$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \vdash e' : A}{\Sigma; \Delta_0 \vdash \texttt{lift}\, e' : \Box(\Gamma \triangleright A)}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{lift}\, e' \xrightarrow{0} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{0} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (ELIFT)/(EERR)

(6) $\Sigma_2; \varnothing \vdash r_2 : A$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) $r_1 = \texttt{box}\, r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and (ELIFT)

(9) $\Sigma_2; \varnothing\varnothing \vdash r_2 : A$ by (6)

(10) $\Sigma_2; \varnothing\Gamma \vdash r_2 : A$ by (9) and weakening

(11) $\Sigma_2; \varnothing \vdash \texttt{box}\, r_2 : \Box(\Gamma \triangleright A)$ by (10) and (TBOX)

Thus, $\Sigma_2; \varnothing \vdash r_1 : \Box(\Gamma \triangleright A)$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (11), (8) and (7)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e' : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{lift}\, e' : \Box(\Gamma \triangleright A)}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{lift}\, e' \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{n} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (ELIFT)/(EERR)

(6) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_2 : A$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) $r_1 = \mathtt{box}\, r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and (ELIFT)

(9) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash \mathtt{lift}\, r_1 : \Box(\Gamma \triangleright A)$ by (6) and (TLIFT)

Thus, $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_1 : \Box(\Gamma \triangleright A)$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (9), (8) and (7)

Case (TGENSYM): Let $e = \lambda^* x.e'$.

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\begin{array}{c}\Sigma; \Delta_0 \cdots \Delta_n + w : A_1 \vdash [x^n \overset{n}{\mapsto} w]\, e' : A_2 \\ w \text{ not in } (\Sigma, \Delta_0 \cdots \Delta_n, \lambda^* x.e')\end{array}}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \lambda^* x.e' : A_1 \to A_2}$
by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \oplus \{w'\} \vdash \lambda^* x.e' \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$
by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \lambda w'.([x^n \overset{n}{\mapsto} w']\, e') \xrightarrow{n} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ by (4) and (EGENSYM)/(EERR)

(6) w' is not in $(\Sigma, \Delta_0 \cdots \Delta_n, \lambda^* x.e')$ by w' is an internal program variable

(7) $\Sigma; \Delta_0 \cdots \Delta_n + w' : A_1 \vdash [x^n \overset{n}{\mapsto} w']\, e' : A_2$ by the premises of (3) and (6)

(8) $\Sigma; \Delta_0 \cdots \Delta_n \vdash \lambda w'.([x^n \overset{n}{\mapsto} w']\, e') : A_1 \to A_2$ by (7) and (TABS)

(9) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_2 : A_1 \to A_2$ and

(10) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (1), (2), (8), (5) and ind. hyp.

(11) $r_1 = r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (9), (10) and (EGENSYM)

Thus, $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_1 : A_1 \to A_2$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (9), (11) and (10)

Case (TREF): Let $e = \mathtt{ref}\, e'$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \vdash e' : A}{\Sigma; \Delta_0 \vdash \mathtt{ref}\, e' : A\, \mathtt{ref}}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \mathtt{ref}\, e' \xrightarrow{0} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{0} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (EREF)/(EERR)

(6) $\Sigma_2; \varnothing \vdash r_2 : A$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) $\mathcal{S}_1 = \mathcal{S}_2 + l : r_2$ for some $l \notin dom(\mathcal{S}_2)$, $r_1 = l$ and $\mathcal{V}_1 = \mathcal{V}_2$
by (4), (5), (6), (7) and (EREF)

(9) $\models (\mathcal{S}_2 + l : r_2) : (\Sigma_2 + l : A)$ by (6), (7) and (8)

(10) $\models \mathcal{S}_1 : \Sigma_1$ where $\Sigma_1 = \Sigma_2 + l : A$ by (8) and (9)

(11) $\Sigma_1; \varnothing \vdash l : A\, \mathtt{ref}$ by (10) and (TSLOC)

Thus, $\Sigma_1; \varnothing \vdash r_1 : A\,\texttt{ref}$ and $\models \mathcal{S}_1 : \Sigma_1$ for some $\Sigma_1 \supseteq \Sigma$
by (11), (8), (10) and (7)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e' : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{ref}\, e' : A\,\texttt{ref}}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{ref}\, e' \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{n} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (EREF)/(EERR)

(6) $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n \vdash r_2 : A$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) $\mathcal{S}_1 = \mathcal{S}_2$, $r_1 = \texttt{ref}\, r_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and (EREF)

(9) $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n \vdash \texttt{ref}\, r_2 : A\,\texttt{ref}$ by (6) and (TREF)

Thus, $\Sigma_2; \varnothing \Delta_1 \cdots \Delta_n \vdash r_1 : A\,\texttt{ref}$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (9), (8) and (7)

Case (TDEREF): Let $e = !\, e'$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \vdash e' : A\,\texttt{ref}}{\Sigma; \Delta_0 \vdash !\, e' : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash !\, e' \xrightarrow{0} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{0} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (EDEREF)/(EERR)

(6) $\Sigma_2; \varnothing \vdash r_2 : A\,\texttt{ref}$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) r_2 is in $V^0 \oplus \{\texttt{err}\}$ by (5) and Lemma 4.2

(9) $r_2 = l$ for some $l \in \mathit{dom}(\mathcal{S}_2)$ by (6), (8) and the def. of V^n

(10) $\Sigma_2; \varnothing \vdash \mathcal{S}_2(l) : A$ by (9), (6) and inv. of (TLOC)

(11) $r_1 = \mathcal{S}_2(l)$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7), (9) and (EDEREF)

Thus, $\Sigma_2; \varnothing \vdash\ r_1 : A$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (11), (10) and (7)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A\,\texttt{ref}}{\Sigma; \Delta_0 \cdots \Delta_n \vdash !\, e : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash !\, e' \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e' \xrightarrow{n} (r_2, \mathcal{S}_2, \mathcal{V}_2)$ for some r_2, $\mathcal{S}_2$ and $\mathcal{V}_2$
by (4) and (EDEREF)/(EERR)

(6) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_2 : A\,\mathtt{ref}$ and

(7) $\models \mathcal{S}_2 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$
by (1), (2), the premise of (3), (5) and ind. hyp.

(8) $r_1 =! r_2$, $\mathcal{S}_1 = \mathcal{S}_2$ and $\mathcal{V}_1 = \mathcal{V}_2$ by (4), (5), (6), (7) and (EDEREF)

(9) $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash\! ! r_2 : A$ by (6) and (TDEREF)

Thus, $\Sigma_2; \varnothing\Delta_1 \cdots \Delta_n \vdash r_1 : A$ and $\models \mathcal{S}_1 : \Sigma_2$ for some $\Sigma_2 \supseteq \Sigma$ by (9), (8) and (7)

Case (TASSIGN): Let $e = e_1 := e_2$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \vdash e_1 : A\,\mathtt{ref} \quad \Sigma; \Delta_0 \vdash e_2 : A}{\Sigma; \Delta_0 \vdash e_1 := e_2 : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 := e_2 \xrightarrow{0} (r_3, \mathcal{S}_3, \mathcal{V}_3)$ for some r_3, $\mathcal{S}_3$ and $\mathcal{V}_3$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{0} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4) and (EASSIGN)/(EERR)

(6) $\Sigma_4; \varnothing \vdash r_4 : A\,\mathtt{ref}$ and

(7) $\models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$
by (1), (2), the first premise of (3), (5) and ind. hyp.

(8) r_4 is in $V^0 \oplus \{\mathtt{err}\}$ by (5) and Lemma 4.2

(9) $r_4 = l$ for some $l \in dom(\mathcal{S}_4)$ by (6), (8) and the def. of V^0

(10) $\mathcal{E}, \mathcal{S}_4, \mathcal{V}_4 \vdash e_2 \xrightarrow{0} (r_5, \mathcal{S}_5, \mathcal{V}_4)$ for some r_5, $\mathcal{S}_5$ and $\mathcal{V}_4$
by (4), (5), (6), (7), (9) and (EASSIGN)/(EERR)

(11) $\Sigma_4; \Delta_0 \vdash e_2 : A$ by the second premise of (3), (7) and Lemma A.1

(12) $\Sigma_4 \models \mathcal{E} : \Delta_0$ by (2), (7) and Lemma A.2

(13) $\Sigma_5; \varnothing \vdash r_5 : A$ and

(14) $\models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma_4$ by (7), (12), (11), (10) and ind. hyp.

(15) $r_3 = r_5$, $\mathcal{S}_3 = \mathcal{S}_5 + l : r_5$ and $\mathcal{V}_3 = \mathcal{V}_5$
by (4), (5), (6), (7), (9), (10), (13), (14) and (EASSIGN)

(16) $\Sigma_5; \varnothing \vdash l : A\,\mathtt{ref}$ by (6), (9), (14) and Lemma A.1

(17) $\Sigma_5(l) = A$ by (16) and inv. of (TLOC)

(18) $\Sigma_5; \varnothing \vdash \mathcal{S}_5(l) : A$ by (14) and (17)

(19) $\Sigma_5; \varnothing \vdash (\mathcal{S}_5 + l : r_5)(l) : A$ by (13) and (18)

(20) $\models (\mathcal{S}_5 + l : r_5) : \Sigma_5$ by (14), (18) and (19)

(21) $\models \mathcal{S}_3 : \Sigma_5$ by (15) and (20)

Thus, $\Sigma_5; \varnothing \vdash r_3 : A$ and $\models \mathcal{S}_3 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma$
by (15), (13), (21), (14) and (7)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 : A\ \mathtt{ref} \quad \Sigma; \Delta_0 \cdots \Delta_n \vdash e_2 : A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 := e_2 : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 := e_2 \xrightarrow{n} (r_3, \mathcal{S}_3, \mathcal{V}_3)$ for some r_3, $\mathcal{S}_3$ and $\mathcal{V}_3$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{n} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4) and `(EASSIGN)`/`(EERR)`

(6) $\Sigma_4; \varnothing \Delta_1 \cdots \Delta_n \vdash r_4 : A\ \mathtt{ref}$ and

(7) $\models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$
by (1), (2), the first premise of (3), (5) and ind. hyp.

(8) $\mathcal{E}, \mathcal{S}_4, \mathcal{V}_4 \vdash e_2 \xrightarrow{n} (r_5, \mathcal{S}_5, \mathcal{V}_5)$ for some r_5, $\mathcal{S}_5$ and $\mathcal{V}_5$
by (4), (5), (6), (7) and `(EASSIGN)`/`(EERR)`

(9) $\Sigma_4; \Delta_0 \cdots \Delta_n \vdash e_2 : A$ by the second premise of (3), (7) and Lemma A.1

(10) $\Sigma_4 \models \mathcal{E} : \Delta_0$ by (2), (7) and Lemma A.2

(11) $\Sigma_5; \varnothing \Delta_1 \cdots \Delta_n \vdash r_5 : A$ and

(12) $\models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma_4$ by (7), (10), (9), (8) and ind. hyp.

(13) $r_3 = r_4 := r_5$, $\mathcal{S}_3 = \mathcal{S}_5$ and $\mathcal{V}_3 = \mathcal{V}_5$
by (4), (5), (6), (7), (8), (11), (12) and `(EASSIGN)`

(14) $\Sigma_5; \varnothing \Delta_1 \cdots \Delta_n \vdash r_4 : A\ \mathtt{ref}$ by (6), (12), and Lemma A.1

(15) $\Sigma_5; \varnothing \Delta_1 \cdots \Delta_n \vdash r_4 := r_5 : A$ by (11), (14) and `(TASSIGN)`

Thus, $\Sigma_5; \varnothing \Delta_1 \cdots \Delta_n \vdash r_3 : A$ and $\models \mathcal{S}_3 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma$
by (15), (13), (12) and (7)

Case `(TLOC)`: Let $e = l$.

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma(l) = A}{\Sigma; \Delta_0 \cdots \Delta_n \vdash l : A\ \mathtt{ref}}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash l \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$ by assumption

(5) $r_1 = l$, $\mathcal{S}_1 = \mathcal{S}$ and $\mathcal{V}_1 = \mathcal{V}$ by (4) and `(ELOC)`

(6) $\Sigma; \varnothing \Delta_1 \cdots \Delta_n \vdash l : A\ \mathtt{ref}$ by the premise of (3) and `(TLOC)`

Thus, $\Sigma; \varnothing \Delta_1 \cdots \Delta_n \vdash r_1 : A$ and $\models \mathcal{S}_1 : \Sigma$ by (6), (5) and (1)

Case `(TCLOS)`: Let $e = \mathtt{clos}(x, e', \mathcal{E}')$.

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\Sigma \models \mathcal{E}' : \Delta \quad \Sigma; \Delta + x : A_1 \vdash e : A_2}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \mathtt{clos}(x, e', \mathcal{E}') : A_1 \to A_2}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \mathtt{clos}(x, e', \mathcal{E}') \xrightarrow{n} (r_1, \mathcal{S}_1, \mathcal{V}_1)$ for some r_1, $\mathcal{S}_1$ and $\mathcal{V}_1$
by assumption

(5) $r_1 = \mathtt{clos}(x, e', \mathcal{E}')$, $\mathcal{S}_1 = \mathcal{S}$ and $\mathcal{V}_1 = \mathcal{V}$ by (4) and `(ECLOS)`

(6) $\Sigma; \Delta'_0 \cdots \Delta'_n \vdash \mathtt{clos}(x, e', \mathcal{E}') : A_1 \to A_2$ for any $\Delta'_0 \cdots \Delta'_n$
by the premises of (3) and `(TCLOS)`

Thus, $\Sigma; \varnothing \Delta_1 \cdots \Delta_n \vdash r_1 : A$ and $\models \mathcal{S}_1 : \Sigma$ by (6), (5) and (1)

Case (TLETIMP): Let $e = \texttt{let}\ (x\ e_1)\ e_2$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\text{expansive}^0(e_1)\quad \Sigma; \Delta_0 \vdash e_1 : B\quad \Sigma; \Delta_0 + x : B \vdash e_2 : A}{\Sigma; \Delta_0 \vdash \texttt{let}\ (x\ e_1)\ e_2 : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{let}\ (x\ e_1)\ e_2 \xrightarrow{0} (r_3, \mathcal{S}_3, \mathcal{V}_3)$ for some r_3, $\mathcal{S}_3$ and $\mathcal{V}_3$
by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{0} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4) and (ELET)/(EERR)

(6) $\Sigma_4; \varnothing \vdash r_4 : B$ and

(7) $\models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$
by (1), (2), the second premise of (3), (5) and ind. hyp.

(8) $\mathcal{E} + x : r_4, \mathcal{S}_4, \mathcal{V}_4 \vdash e_2 \xrightarrow{0} (r_5, \mathcal{S}_5, \mathcal{V}_5)$ for some r_5, $\mathcal{S}_5$ and $\mathcal{V}_5$
by (4), (5), (6), (7) and (ELET)/(EERR)

(9) $\Sigma_4 \models \mathcal{E} : \Delta_0$ by (2), (7) and Lemma A.2

(10) $\Sigma_4 \models (\mathcal{E} + x : r_4) : (\Delta_0 + x : B)$ by (9) and (6)

(11) $\Sigma_4; \Delta_0 + x : B \vdash e_2 : A$ by the third premise of (3), (7) and Lemma A.1

(12) $\Sigma_5; \varnothing \vdash r_5 : A$ and

(13) $\models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma_4$ by (7), (10), (11), (8) and ind. hyp.

(14) $r_3 = r_5$, $\mathcal{S}_5 = \mathcal{S}_3$ and $\mathcal{V}_5 = \mathcal{V}_3$
by (4), (5), (6), (7), (8), (12), (13) and (ELET)

Thus, $\Sigma_5; \varnothing \vdash r_3 : A$ and $\models \mathcal{S}_3 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma$ by (14), (12), (13) and (7)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\begin{array}{l}\text{expansive}^n(e_1)\\ \Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 : B\\ \Sigma; \Delta_0 \cdots \Delta_n + x : B \vdash e_2 : A\end{array}}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \texttt{let}\ (x\ e_1)\ e_2 : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{let}\ (x\ e_1)\ e_2 \xrightarrow{n} (r_3, \mathcal{S}_3, \mathcal{V}_3)$ for some r_3, $\mathcal{S}_3$ and $\mathcal{V}_3$
by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{n} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4) and (ELET)/(EERR)

(6) $\Sigma_4; \varnothing \Delta_1 \cdots \Delta_n \vdash r_4 : B$ and

(7) $\models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$
by (1), (2), the second premise of (3), (5) and ind. hyp.

(8) $\mathcal{E}, \mathcal{S}_4, \mathcal{V}_4 \vdash e_2 \xrightarrow{n} (r_5, \mathcal{S}_5, \mathcal{V}_5)$ for some r_5, $\mathcal{S}_5$ and $\mathcal{V}_5$
by (4), (5), (6), (7) and (ELET)/(EERR)

(9) $\Sigma_4 \models \mathcal{E} : \Delta_0$ by (2), (7) and Lemma A.2

(10) $\Sigma_4; \Delta_0 \cdots \Delta_n + x : B \vdash e_2 : A$
by the third premise of (3), (7) and Lemma A.1

(11) $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n + x : B \vdash r_5 : A$ and

(12) $\quad \models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma_4$ by (7), (9), (10), (8) and ind. hyp.

(13) $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n \vdash r_4 : B$ by (6), (12) and Lemma A.1

(14) $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n \vdash \texttt{let}\ (x\ r_4)\ r_5 : A$
by the first premise of (3), (13), (11) and `(TLETIMP)`

(15) $r_3 = \texttt{let}\ (x\ r_4)\ r_5$, $\mathcal{S}_3 = \mathcal{S}_5$ and $\mathcal{V}_3 = \mathcal{V}_5$
by (4), (5), (6), (7), (8), (11), (12) and `(ELET)`

Thus, $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n \vdash r_3 : A$ and $\models \mathcal{S}_3 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma$
by (15), (14), (12) and (7)

Case `(TLETAPP)`: Let $e = \texttt{let}\ (x\ e_1)\ e_2$.
If $n = 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $\dfrac{\begin{array}{l}\neg\,\text{expansive}^0(e_1) \\ \Sigma; \Delta_0 \vdash e_1 : B \\ \Sigma; \Delta_0 + x : GEN_B(\Sigma, \Delta_0) \vdash e_2 : A\end{array}}{\Sigma; \Delta_0 \vdash \texttt{let}\ (x\ e_1)\ e_2 : A}$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \texttt{let}\ (x\ e_1)\ e_2 \xrightarrow{0} (r_3, \mathcal{S}_3, \mathcal{V}_3)$ for some r_3, $\mathcal{S}_3$ and $\mathcal{V}_3$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{0} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4) and `(ELET)`/`(EERR)`

(6) $\Sigma_4; \varnothing \vdash r_4 : B$ and

(7) $\quad \models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$
by (1), (2), the second premise of (3), (5) and ind. hyp.

(8) Let $GEN_B(\Sigma, \Delta_0) = \forall\xi_1 \ldots \xi_k.B$ such that

(9) $\quad \{\xi_1, \ldots, \xi_k\} = \text{FV}(B) \setminus (\text{FV}(\Delta_0) \cup \text{FV}(\Sigma))$

(10) Let B' be any type satisfying $GEN_B(\Sigma, \Delta_0) \succ B'$.

(11) There exists a substitution R such that

(12) $\quad RB = B'$ and $dom(R) = \{\xi_1, \ldots, \xi_k\}$ by (8), (9) and (10)

(13) $R\Sigma_4; \varnothing \vdash r_4 : RB$ by (6), (11) and Lemma A.3

(14) $\Sigma_4 = \Sigma$ by $\neg\,\text{expansive}^0(e_1)$

(15) $\Sigma; \varnothing \vdash r_4 : B'$ by (13), (14), (12) and (9)

(16) $\Sigma \models (\mathcal{E} + x : r_4) : (\Delta_0 + x : GEN_B(\Sigma, \Delta_0))$ by (2), (10) and (15)

(17) $\mathcal{E} + x : r_4, \mathcal{S}_4, \mathcal{V}_4 \vdash e_2 \xrightarrow{n} (r_5, \mathcal{S}_5, \mathcal{V}_5)$ for some r_5, $\mathcal{S}_5$ and $\mathcal{V}_5$
by (4), (5), (6), (7) and `(ELET)`/`(EERR)`

(18) $\Sigma_5; \varnothing \vdash r_5 : A$ and

(19) $\quad \models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma_4$
by (1), (16), the third premise of (3), (14), (17) and ind. hyp.

(20) $r_3 = r_5$, $\mathcal{S}_3 = \mathcal{S}_5$ and $\mathcal{V}_3 = \mathcal{V}_5$
by (4), (5), (6), (7), (17), (18), (19) and `(ELET)`

Thus, $\Sigma_5; \varnothing \vdash r_3 : A$ and $\models \mathcal{S}_3 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma$
by (20), (18), (19) and (7)

If $n > 0$, *then*

(1) $\models \mathcal{S} : \Sigma$ by assumption

(2) $\Sigma \models \mathcal{E} : \Delta_0$ by assumption

(3) $$\frac{\begin{array}{l}\neg\,\mathrm{expansive}^n(e_1)\\ \Sigma; \Delta_0 \cdots \Delta_n \vdash e_1 : B\\ \Sigma; \Delta_0 \cdots \Delta_n + x : GEN_B(\Sigma, \Delta_0 \cdots \Delta_n) \vdash e_2 : B\end{array}}{\Sigma; \Delta_0 \cdots \Delta_n \vdash \mathtt{let}\ (x\ e_1)\ e_2 : A}$$ by assumption

(4) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash \mathtt{let}\ (x\ e_1)\ e_2 \xrightarrow{n} (r_3, \mathcal{S}_3, \mathcal{V}_3)$ for some r_3, $\mathcal{S}_3$ and $\mathcal{V}_3$ by assumption

(5) $\mathcal{E}, \mathcal{S}, \mathcal{V} \vdash e_1 \xrightarrow{n} (r_4, \mathcal{S}_4, \mathcal{V}_4)$ for some r_4, $\mathcal{S}_4$ and $\mathcal{V}_4$
by (4) and `(ELET)`/`(EERR)`

(6) $\Sigma_4; \varnothing\Delta_1 \cdots \Delta_n \vdash r_4 : B$ and

(7) $\models \mathcal{S}_4 : \Sigma_4$ for some $\Sigma_4 \supseteq \Sigma$
by (1), (2), the second premise of (3), (5) and ind. hyp.

(8) $\mathcal{E}, \mathcal{S}_4, \mathcal{V}_4 \vdash e_2 \xrightarrow{n} (r_5, \mathcal{S}_5, \mathcal{V}_5)$ for some r_5, $\mathcal{S}_5$ and $\mathcal{V}_5$
by (4), (5), (6), (7) and `(ELET)`/`(EERR)`

(9) $\Sigma_4 = \Sigma$ by $\neg\,\mathrm{expansive}^n(e_1)$

(10) $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n + x : GEN_B(\Sigma, \Delta_0 \cdots \Delta_n) \vdash r_5 : A$ and

(11) $\models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma_4$
by (9), (1), (2), the third premise of (3), (8) and ind. hyp.

(12) Let $GEN_B(\Sigma, \Delta_0 \cdots \Delta_n) = \forall \xi_1 \ldots \xi_k.B$ such that

(13) $\{\xi_1, \ldots, \xi_k\} = \mathrm{FV}(B) \setminus (\mathrm{FV}(\Sigma) \cup \bigcup_{i=0}^{n} \mathrm{FV}(\Delta_i))$.

(14) There exists a bijective substitution $R = \{\xi_i : \chi_i\}_1^k$ such that

(15) $\mathit{range}(R) \cap (\mathrm{FV}(B) \cup \mathrm{FV}(\Sigma_5) \cup \bigcup_{i=0}^{n} \mathrm{FV}(\Delta_i)) = \varnothing$.

(16) $R\Sigma_4; \varnothing(R\Delta_1) \cdots (R\Delta_n) \vdash r_4 : RB$ by (6), (14) and Lemma A.3

(17) $\Sigma; \varnothing\Delta_1 \cdots \Delta_n \vdash r_4 : RB$ by (16), (9), (13) and (14)

(18) $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n \vdash r_4 : RB$ by (17), (11), (9) and Lemma A.1

(19) $GEN_B(\Sigma, \Delta_0 \cdots \Delta_n)$
$= \forall \xi_1 \ldots \xi_k.B$ by (12) and (13)
$= \forall \chi_1 \ldots \chi_k.(RB)$ by (12), (13), (14) and (15)
$= GEN_{RB}(\Sigma_5, \Delta_0 \cdots \Delta_n)$ by (15) and the def. of *GEN*

(20) $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n + x : GEN_{RB}(\Sigma_5, \Delta_0 \cdots \Delta_n) \vdash r_5 : A$ by (10) and (19)

(21) For any A, if $GEN_{RB}(\Sigma_5, \Delta_0 \cdots \Delta_n) \succ A$, then $GEN_{RB}(\Sigma_5, \varnothing\Delta_1 \cdots \Delta_n) \succ A$.

(22) $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n + x : GEN_{RB}(\Sigma_5, \varnothing\Delta_1 \cdots \Delta_n) \vdash r_5 : A$ by (20) and (21)

(23) $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n \vdash \mathtt{let}\ (x\ r_4)\ r_5 : A$
by the first premise of (3), (18), (22) and `(TLETAPP)`

(24) $r_3 = \mathtt{let}\ (x\ r_4)\ r_5$, $\mathcal{S}_3 = \mathcal{S}_5$ and $\mathcal{V}_3 = \mathcal{V}_5$
by (4), (5), (6), (7), (8), (10), (11) and `(ELET)`

Thus, $\Sigma_5; \varnothing\Delta_1 \cdots \Delta_n \vdash r_3 : A$ and $\models \mathcal{S}_5 : \Sigma_5$ for some $\Sigma_5 \supseteq \Sigma$
by (24), (23), (11) and (9)

□

Lemma A.1 If $\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$ and $\Sigma \subseteq \Sigma'$, then $\Sigma'; \Delta_0 \cdots \Delta_n \vdash e : A$.

PROOF By induction on the type derivation of $\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$. For a given type derivation, we proceed by cases on the finally used type rule.

□

Lemma A.2 If $\Sigma \models \mathcal{E} : \Delta$ and $\Sigma \subseteq \Sigma'$, then $\Sigma' \models \mathcal{E} : \Delta$.

PROOF From the definition of $\Sigma \models \mathcal{E} : \Delta$, $dom(\mathcal{E}) = dom(\Delta)$ and

$$\Sigma; \varnothing \vdash \mathcal{E}(x) : A$$

for every $(x, A) \in \{(x, A) \mid x \in dom(\Delta), \Delta(x) \succ A\}$. From $\Sigma \subseteq \Sigma'$ and Lemma A.1,

$$\Sigma'; \varnothing \vdash \mathcal{E}(x) : A$$

for every $(x, A) \in \{(x, A) \mid x \in dom(\Delta), \Delta(x) \succ A\}$. Thus, $\Sigma' \models \mathcal{E} : \Delta$. □

Lemma A.3 If $\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$ and S is a substitution, then

$$S\Sigma; (S\Delta_0) \cdots (S\Delta_n) \vdash e : SA.$$

PROOF By induction on the type derivation of $\Sigma; \Delta_0 \cdots \Delta_n \vdash e : A$. For a given type derivation, we proceed by cases on the finally used type rule.

□

Lemma A.4 Assume $S\mu \succ A$ and V is a set of type variables with $\mathrm{FV}(\mu) \subseteq V$. Then there exists a type A' and a substitution S' such that $\mu \succ A'$, $S'A' = A$ and $S|_V = S'|_V$.

PROOF The proof of this lemma is similar to Lemma 5.3 in [24]. □

Lemma A.5 For any S, if $\mu \succ A$, then $S\mu \succ SA$.

PROOF $\mathrm{RV}(S)$ (related variables of S) is defined in Section 5. Let $\mu = \forall \xi_1 \ldots \xi_m.F$. From the assumption, there exists a bijective substitution $R = \{\xi_i : X_i\}_1^m$ satisfying $RF = A$. Then, $S\mu$ is of the form $\forall \chi_1 \ldots \chi_m.(S_0 :: \{\xi_i : \chi_i\}_1^m)\, F$ where $S_0 = S|^-_{\{\xi_1,\ldots,\xi_m\}}$, $\{\chi_i\}_1^m \cap \mathrm{FV}(F) = \varnothing$ and $\{\chi_i\}_1^m \cap \mathrm{RV}(S) = \varnothing$. Then, we will show that

$$\{\chi_i : SX_i\}_1^m\, (S_0 :: \{\xi_i : \chi_i\}_1^m)\, F = SRF,$$

which will be proved by showing that

$$\{\chi_i : SX_i\}_1^m\, (S_0 :: \{\xi_i : \chi_i\}_1^m)\, \xi = SR\xi$$

for each ξ occurring in F where ξ is a type variable, a type environment variable or a field variable.

1. If ξ is a bound variable in μ and $\xi = \xi_j$, then

$$\begin{aligned}
&\{\chi_i : SX_i\}_1^m\, (S_0 :: \{\xi_i : \chi_i\}_1^m)\, \xi_j \\
&\quad = \{\chi_i : SX_i\}_1^m\, \chi_j \\
&\quad = SX_j \\
&\quad = SR\xi_j
\end{aligned}$$

2. If ξ is a free variable in μ, then

$$\begin{aligned}
&\{\chi_i : SX_i\}_1^m\, (S_0 :: \{\xi_i : \chi_i\}_1^m)\, \xi \\
&\quad = \{\chi_i : SX_i\}_1^m\, S_0\xi && \text{since } \xi \notin \{\xi_i\}_1^m \\
&\quad = \{\chi_i : SX_i\}_1^m\, S\xi && \text{since } S_0 = S|^-_{\{\xi_1,\ldots,\xi_m\}} \\
&\quad = S\xi \\
& && \text{since } \{\chi_i\}_1^m \cap \mathrm{RV}(S) = \varnothing \text{ and } \{\chi_i\}_1^m \cap \mathrm{FV}(F) = \varnothing
\end{aligned}$$

$= SR\xi$ since $\xi \notin \{\xi_i\}_1^m$

From (1) and (2),

$$\{\chi_i : SX_i\}_1^m\ (S_0 :: \{\xi_i : \chi_i\}_1^m)\ \xi = SR\xi$$

for each ξ occurring in F. Hence,

$$\{\chi_i : SX_i\}_1^m\ (S_0 :: \{\xi_i : \chi_i\}_1^m)\ F = SRF.$$

From $RF = A$, it follows that $S\mu \succ SA$. □

Lemma A.6 For any S, if $\Delta \succ \Gamma$, then $S\Delta \succ S\Gamma$.

- If $\Delta = \{x_i : \mu_i\}_1^m\rho$, then

  (1) $\Gamma = \{x_i : F_i\}_1^m\rho$ such that $\mu_i \succ F_i$ for $i \in [1..m]$ by $\Delta \succ \Gamma$

  (2) $S\Delta = \{x_i : S\mu_i\}_1^m :: (S\rho)$

  (3) $S\mu_i \succ SF_i$ for $i \in [1..m]$ by (1) and Lemma A.5

  (4) $S\rho \succ S\rho$

  (5) $\{x_i : S\mu_i\}_1^m :: (S\rho) \succ \{x_i : SF_i\}_1^m :: (S\rho)$ by (3) and (4)

  (6) $\{x_i : SF_i\}_1^m :: (S\rho) = S\Gamma$ by (1)

  Thus, $S\Delta \succ S\Gamma$ by (2), (5) and (6)

- If $\Delta = \{x_i : \mu_i\}_1^m$, then $S\Delta \succ S\Gamma$.

  This case is similar to the previous case. We omit the detailed proof.

Lemma 6.1

PROOF By induction on the type derivation of $\Gamma_0 \cdots \Gamma_n \vdash^i e : A$ in the implicit $\lambda^{\Box}$ language. For a given type derivation, we proceed by cases on the finally used type rule.

- Let $e = x$.

  (1) $\dfrac{\Gamma_n(x) = A}{\Gamma_0 \cdots \Gamma_n \vdash^i x : A}$ by assumption

  (2) $[\![\Gamma_n]\!](x) = [\![A]\!]$ by the premise of (1) and translation rule

  Thus, $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash x : [\![A]\!]$ by (2) and `(TSVAR)`

- Let $e = \lambda x.e'$.

  (1) $\dfrac{\Gamma_0 \cdots \Gamma_n + x : A_1 \vdash^i e' : A_2}{\Gamma_0 \cdots \Gamma_n \vdash^i \lambda x.e' : A_1 \to A_2}$ by assumption

  (2) $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] + x : [\![A_1]\!] \vdash [\![e']\!] : [\![A_2]\!]$ by the premise of (1) and ind. hyp.

  Thus, $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash \lambda x.[\![e']\!] : [\![A_1]\!] \to [\![A_2]\!]$ by (2) and `(TSABS)`

- Let $e = e_1 e_2$.

  (1) $\dfrac{\Gamma_0 \cdots \Gamma_n \vdash^i e_1 : B \to A \qquad \Gamma_0 \cdots \Gamma_n \vdash^i e_2 : B}{\Gamma_0 \cdots \Gamma_n \vdash^i e_1 e_2 : A}$ by assumption

  (2) $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash [\![e_1]\!] : [\![B \to A]\!]$ by the first premise of (1) and ind. hyp.

  (3) $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash [\![e_2]\!] : [\![B]\!]$ by the second premise of (1) and ind. hyp.

  (4) $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash [\![e_1]\!] : [\![B]\!] \to [\![A]\!]$ by (2) and translation rule

Thus, $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash [\![e_1]\!][\![e_2]\!] : [\![A]\!]$ by (3), (4) and (TSAPP)

- Let $e = \texttt{box}\, e'$.

(1) $\dfrac{\Gamma_0 \cdots \Gamma_n \varnothing \vdash^i e' : A}{\Gamma_0 \cdots \Gamma_n \vdash^i \texttt{box}\, e' : \Box A}$ by assumption

(2) $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!][\![\varnothing]\!] \vdash [\![e']\!] : [\![A]\!]$ by the premise of (1) and ind. hyp.

(3) $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!]\varnothing \vdash [\![e']\!] : [\![A]\!]$ by (2) and translation rule

(4) $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash \texttt{box}\, [\![e']\!] : \Box(\varnothing \triangleright [\![A]\!])$ by (3) and (TSBOX)

Thus, $[\![\Gamma_0]\!] \cdots [\![\Gamma_n]\!] \vdash \texttt{box}\, [\![e']\!] : [\![\Box A]\!]$ by (4) and translation rule

- Let $e = \texttt{unbox}_k\, e'$.

(1) $\dfrac{\Gamma_0 \cdots \Gamma_{n-k} \vdash^i e' : \Box A}{\Gamma_0 \cdots \Gamma_{n-k} \cdots \Gamma_n \vdash^i \texttt{unbox}_k\, e' : A}$ by assumption

(2) $[\![\Gamma_0]\!] \cdots [\![\Gamma_{n-k}]\!] \vdash [\![e']\!] : [\![\Box A]\!]$ by the premise of (1) and ind. hyp.

(3) $[\![\Gamma_0]\!] \cdots [\![\Gamma_{n-k}]\!] \vdash [\![e']\!] : \Box(\varnothing \triangleright [\![A]\!])$ by (2) and translation rule

(4) $[\![\Gamma_0]\!] \cdots [\![\Gamma_{n-k}]\!] \vdash \texttt{open}\, [\![e']\!] : \Box([\![\Gamma_n]\!] \triangleright [\![A]\!])$ by (3) and (TSOPEN)

Thus, $[\![\Gamma_0]\!] \cdots [\![\Gamma_{n-k}]\!] \cdots [\![\Gamma_n]\!] \vdash \texttt{unbox}_k\, (\texttt{open}\, [\![e']\!]) : [\![A]\!]$ by (4) and (TSUNBOX)

□

Lemma 6.2

PROOF By induction on the type derivation of $\Gamma \vdash^L e : A$ in λ^i without cross-stage persistence. For a given type derivation, we proceed by cases on the finally used type rule.

- Let $e = x$.

(1) $\dfrac{\Gamma(x) = A^L}{\Gamma \vdash^L x : A}$ by assumption

(2) $x : A^L \in \Gamma$ by the premise of (1)

(3) $x : A^L \in \Phi$ by (2) and the def. of CTE($\Gamma \vdash^L e : A$).

(4) $x : \texttt{Tr}(\Phi, A, L) \in [\![\Phi]\!]^L_{|L|}$ by (3)

(5) $[\![\Phi]\!]^L_{|L|}(x) = \texttt{Tr}(\Phi, A, L)$ by (4)

Thus, $[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![x]\!] : \texttt{Tr}(\Phi, A, L)$ by (5) and (TSVAR)

- Let $e = \lambda x.e'$.

(1) $\dfrac{\Gamma + x : A^L \vdash^L e' : B}{\Gamma \vdash^L \lambda x.e' : A \to B}$ by assumption

(2) Let Φ' be any function satisfying $\Phi' \supseteq \mathrm{CTE}(\Gamma + x : A^L \vdash^L e' : B)$.

(3) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} \vdash [\![e']\!] : \texttt{Tr}(\Phi', B, L)$ by the premise of (1), (2) and ind. hyp.

(4) $x : A^L \in \Phi'$ by (2)

(5) $x : \texttt{Tr}(\Phi', A, L) \in [\![\Phi']\!]^L_{|L|}$ by (4)

(6) Let $[\![\Phi']\!]^L_{|L|} = \{x : \texttt{Tr}(\Phi', A, L)\} :: \Gamma'$ for some Γ' by (5)

(7) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|-1} \Gamma' \vdash \lambda x.[\![e']\!] : \texttt{Tr}(\Phi', A, L) \to \texttt{Tr}(\Phi', B, L)$ by (3), (6) and (TSABS)

(8) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|-1} \Gamma' \vdash \lambda^* x.[\![e']\!] : \texttt{Tr}(\Phi', A, L) \to \texttt{Tr}(\Phi', B, L)$ by (7), x or $(x : A^L)$ is unique in Φ' by assumption and (TSGENSYM)

(9) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|-1} \Gamma' + x : \mathtt{Tr}(\Phi', A, L) \vdash \lambda^* x.[\![e']\!] : \mathtt{Tr}(\Phi', A, L) \to \mathtt{Tr}(\Phi', B, L)$
by (8) and x or $(x : A^L)$ is unique in Φ' by assumption

(10) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} \vdash \lambda^* x.[\![e']\!] : \mathtt{Tr}(\Phi', A, L) \to \mathtt{Tr}(\Phi', B, L)$ by (6) and (9)

(11) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} \vdash [\![\lambda x.e']\!] : \mathtt{Tr}(\Phi', A \to B, L)$ by (10) and translation rule

(12) $\mathrm{CTE}(\Gamma \vdash^L \lambda x.e' : B)$
$= \Gamma \cup \mathrm{CTE}(\Gamma + x : A^L \vdash^L e' : B)$ by the def. of CTE
$= \mathrm{CTE}(\Gamma + x : A^L \vdash^L e' : B)$
since $\Gamma \subseteq \mathrm{CTE}(\Gamma + x : A^L \vdash^L e' : B)$

Thus, for any function $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^L \lambda x.e' : B)$,
$[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![\lambda x.e']\!] : \mathtt{Tr}(\Phi, A \to B, L)$ by (2), (11) and (12)

• Let $e = e_1 e_2$.

(1) $\dfrac{\Gamma \vdash^L e_1 : B \to A \quad \Gamma \vdash^L e_2 : B}{\Gamma \vdash^L e_1 e_2 : A}$ by assumption

(2) Let Φ_1 be any function satisfying $\Phi_1 \supseteq \mathrm{CTE}(\Gamma \vdash^L e_1 : B \to A)$.

(3) $[\![\Phi_1]\!]^L_0 \cdots [\![\Phi_1]\!]^L_{|L|} \vdash [\![e_1]\!] : \mathtt{Tr}(\Phi_1, B \to A, L)$
by the first premise of (1), (2) and ind. hyp.

(4) Let Φ_2 be any function satisfying $\mathrm{CTE}(\Gamma \vdash^L e_2 : B)$.

(5) $[\![\Phi_2]\!]^L_0 \cdots [\![\Phi_2]\!]^L_{|L|} \vdash [\![e_2]\!] : \mathtt{Tr}(\Phi_2, B, L)$
by the second premise of (1) and ind. hyp.

(6) $\mathrm{CTE}(\Gamma \vdash^L e_1 e_2 : A)$
$= \Gamma \cup \mathrm{CTE}(\Gamma \vdash^L e_1 : B \to A) \cup \mathrm{CTE}(\Gamma \vdash^L e_2 : B)$ by the def. of CTE
$= \mathrm{CTE}(\Gamma \vdash^L e_1 : B \to A) \cup \mathrm{CTE}(\Gamma \vdash^L e_2 : B)$
since $\Gamma \subseteq \mathrm{CTE}(\Gamma \vdash^L e_1 : B \to A) \cup \mathrm{CTE}(\Gamma \vdash^L e_2 : B)$

(7) Let Φ be any function satisfying $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^L e_1 e_2 : A)$.

(8) $[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![e_1]\!] : \mathtt{Tr}(\Phi, B \to A, L)$ by (2), (3), (6) and (7)

(9) $[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![e_2]\!] : \mathtt{Tr}(\Phi, B, L)$ by (4), (5), (6) and (7)

(10) $[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![e_1]\!] : \mathtt{Tr}(\Phi, B, L) \to \mathtt{Tr}(\Phi, A, L)$ by (8) and translation rule

Thus, $[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![e_1 e_2]\!] : \mathtt{Tr}(\Phi, A, L)$ by (9), (10) and `(TSAPP)`

• Let $e = {}^\sim e'$.

(1) $\dfrac{\Gamma \vdash^L e' : \langle A \rangle^\alpha}{\Gamma \vdash^{L\alpha} {}^\sim e' : A}$ by assumption

(2) Let Φ' be any function satisfying $\Phi' \supseteq \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle^\alpha)$.

(3) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} \vdash [\![e']\!] : \mathtt{Tr}(\Phi', \langle A \rangle^\alpha, L)$
by the premise of (1), (2) and ind. hyp.

(4) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} \vdash [\![e']\!] : \Box(\Phi'^{L\alpha}_{|L\alpha|} \triangleright \mathtt{Tr}(\Phi', A, L\alpha))$ by (3) and translation rule

(5) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} [\![\Phi']\!]^{L\alpha}_{|L\alpha|} \vdash \mathtt{unbox}_1\, [\![e']\!] : \mathtt{Tr}(\Phi', A, L\alpha)$ by (4) and `(TSUNBOX)`

(6) $[\![\Phi']\!]^{L\alpha}_0 \cdots [\![\Phi']\!]^{L\alpha}_{|L|} [\![\Phi']\!]^{L\alpha}_{|L\alpha|} \vdash \mathtt{unbox}_1\, [\![e']\!] : \mathtt{Tr}(\Phi', A, L\alpha)$
by (5) and translation rule

(7) $\mathrm{CTE}(\Gamma \vdash^{L\alpha} {}^\sim e' : A)$
$= \Gamma \cup \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle^\alpha)$ by the def. of CTE
$= \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle^\alpha)$ since $\Gamma \subseteq \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle^\alpha)$

Thus, for any function $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^{L\alpha} \tilde{}\, e' : A)$,
$[\![\Phi]\!]^{L\alpha}_0 \cdots [\![\Phi]\!]^{L\alpha}_{|L|} [\![\Phi]\!]^{L\alpha}_{|L\alpha|} \vdash [\![\tilde{}\, e']\!] : \mathtt{Tr}(\Phi, A, L\alpha)$
by (2), (6), (7) and translation rule

- Let $e = \langle e' \rangle$.

(1) $\dfrac{\Gamma \vdash^{L\alpha} e' : A}{\Gamma \vdash^{L} \langle e' \rangle : \langle A \rangle^{\alpha}}$ by assumption

(2) Let Φ' be any function satisfying $\Phi' \supseteq \mathrm{CTE}(\Gamma \vdash^{L\alpha} e' : A)$.

(3) $[\![\Phi']\!]^{L\alpha}_0 \cdots [\![\Phi']\!]^{L\alpha}_{|L\alpha|-1} [\![\Phi']\!]^{L\alpha}_{|L\alpha|} \vdash [\![e']\!] : \mathtt{Tr}(\Phi', A, L\alpha)$
by the premise of (1), (2) and ind. hyp.

(4) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} [\![\Phi']\!]^{L\alpha}_{|L\alpha|} \vdash [\![e']\!] : \mathtt{Tr}(\Phi', A, L\alpha)$ by (3) and translation rule

(5) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash \mathtt{box}\, [\![e']\!] : \Box([\![\Phi']\!]^{L\alpha}_{|L\alpha|} \triangleright \mathtt{Tr}(\Phi', A, L\alpha))$ by (4) and `(TSBOX)`

(6) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash [\![\langle e' \rangle]\!] : \mathtt{Tr}(\Phi', \langle A \rangle^{\alpha}, L)$ by (5) and translation rule

(7) $\mathrm{CTE}(\Gamma \vdash^{L} \langle e' \rangle : \langle A \rangle^{\alpha})$
$= \Gamma \cup \mathrm{CTE}(\Gamma \vdash^{L\alpha} e' : A)$ by the def. of CTE
$= \mathrm{CTE}(\Gamma \vdash^{L\alpha} e' : A)$ since $\Gamma \subseteq \mathrm{CTE}(\Gamma \vdash^{L\alpha} e' : A)$

Thus, for any function satisfying $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^{L} \langle e' \rangle : \langle A \rangle^{\alpha})$,
$[\![\Phi]\!]^{L}_0 \cdots [\![\Phi]\!]^{L}_{|L|} \vdash [\![\langle e' \rangle]\!] : \mathtt{Tr}(\Phi, \langle A \rangle^{\alpha}, L)$ by (2), (6) and (7)

- Let $e = \mathtt{open}\ e'$.

(1) $\dfrac{\Gamma \vdash^{L} e' : \langle A \rangle}{\Gamma \vdash^{L} \mathtt{open}\ e' : \langle A \rangle^{\alpha}}$ by assumption

(2) Let Φ' be any function satisfying $\Phi' \supseteq \mathrm{CTE}(\Gamma \vdash^{L} e' : \langle A \rangle)$.

(3) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash [\![e']\!] : \mathtt{Tr}(\Phi', \langle A \rangle, L)$ by the premise of (1), (2) and ind. hyp.

(4) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash [\![e']\!] : \Box(\varnothing \triangleright \mathtt{Tr}(\Phi', A, L\alpha))$ by (3) and translation rule

(5) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash \mathtt{open}\, [\![e']\!] : \Box([\![\Phi']\!]^{L\alpha}_{|L\alpha|} \triangleright \mathtt{Tr}(\Phi', A, L\alpha))$ by (4) and `(TSOPEN)`

(6) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash [\![\mathtt{open}\, e']\!] : \mathtt{Tr}(\Phi', \langle A \rangle^{\alpha}, L)$ by (5) and translation rule

(7) $\mathrm{CTE}(\Gamma \vdash^{L} \mathtt{open}\, e' : \langle A \rangle^{\alpha})$
$= \Gamma \cup \mathrm{CTE}(\Gamma \vdash^{L} e' : \langle A \rangle)$ by the def. of CTE
$= \mathrm{CTE}(\Gamma \vdash^{L} e' : \langle A \rangle)$ since $\Gamma \subseteq \mathrm{CTE}(\Gamma \vdash^{L} e' : \langle A \rangle)$

Thus, for any function $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^{L} \mathtt{open}\, e' : \langle A \rangle^{\alpha})$,
$[\![\Phi]\!]^{L}_0 \cdots [\![\Phi]\!]^{L}_{|L|} \vdash [\![\mathtt{open}\, e']\!] : \mathtt{Tr}(\Phi, \langle A \rangle^{\alpha}, L)$ by (2), (6) and (7)

- Let $e = \mathtt{close}\, e'$.

(1) $\dfrac{\Gamma \vdash^{L} e' : \langle A \rangle^{\alpha} \quad \alpha \notin \mathrm{FV}(\Gamma, L, A)}{\Gamma \vdash^{L} \mathtt{close}\, e' : \langle A \rangle}$ by assumption

(2) Let Φ' be any function satisfying $\Phi' \supseteq \mathrm{CTE}(\Gamma \vdash^{L} \mathtt{close}\, e' : \langle A \rangle)$.

(3) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash [\![e']\!] : \mathtt{Tr}(\Phi', \langle A \rangle^{\alpha}, L)$
by the first premise of (1), (2) and ind. hyp.

(4) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash [\![e']\!] : \Box([\![\Phi']\!]^{L\alpha}_{|L\alpha|} \triangleright \mathtt{Tr}(\Phi', A, L\alpha))$ by (3) and translation rule

(5) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash [\![e']\!] : \Box(\varnothing \triangleright \mathtt{Tr}(\Phi', A, L\alpha))$
by the second premise of (1) and (4)

(6) $[\![\Phi']\!]^{L}_0 \cdots [\![\Phi']\!]^{L}_{|L|} \vdash [\![\mathtt{close}\, e']\!] : \mathtt{Tr}(\Phi', \langle A \rangle, L)$ by (5) and translation rule

(7) $\mathrm{CTE}(\Gamma \vdash^L \texttt{close}\, e' : \langle A \rangle)$
$= \Gamma \cup \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle^\alpha)$ by the def. of CTE
$= \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle^\alpha)$ since $\Gamma \subseteq \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle^\alpha)$

Thus, for any function $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^L \texttt{close}\, e' : \langle A \rangle)$,
$[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![\texttt{close}\, e']\!] : \texttt{Tr}(\Phi, \langle A \rangle, L)$ by (2), (6) and (7)

- Let $e = \texttt{run}\, e'$.

(1) $\dfrac{\Gamma \vdash^L e' : \langle A \rangle}{\Gamma \vdash^L \texttt{run}\, e' : A}$ by assumption

(2) Let Φ' be any function satisfying $\Phi' \supseteq \mathrm{CTE}(\Gamma \vdash^L \texttt{run}\, e' : A)$.

(3) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_n \vdash [\![e']\!] : \texttt{Tr}(\Phi', \langle A \rangle, L)$ by the premise of (1), (2) and ind. hyp.

(4) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_n \vdash [\![e']\!] : \Box(\varnothing \triangleright \texttt{Tr}(\Phi', A, L))$ by (3) and translation rule

(5) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} \vdash \texttt{open}\, [\![e']\!] : \Box([\![\Phi']\!]^L_{|L|} \triangleright \texttt{Tr}(\Phi', A, L))$ by (4) and `(TSOPEN)`

(6) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} \vdash \texttt{unbox}_0\, (\texttt{open}\, [\![e']\!]) : \texttt{Tr}(\Phi', A, L)$ by (5) and `(TSUNBOX)`

(7) $[\![\Phi']\!]^L_0 \cdots [\![\Phi']\!]^L_{|L|} \vdash [\![\texttt{run}\, e']\!] : \texttt{Tr}(\Phi', A, L)$ by (6) and translation rule

(8) $\mathrm{CTE}(\Gamma \vdash^L \texttt{run}\, e' : A)$
$= \Gamma \cup \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle)$ by the def. of CTE
$= \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle)$ since $\Gamma \subseteq \mathrm{CTE}(\Gamma \vdash^L e' : \langle A \rangle)$

Thus, for any function $\Phi \supseteq \mathrm{CTE}(\Gamma \vdash^L \texttt{run}\, e' : A)$,
$[\![\Phi]\!]^L_0 \cdots [\![\Phi]\!]^L_{|L|} \vdash [\![\texttt{run}\, e']\!] : \texttt{Tr}(\Phi, A, L)$ by (2), (7) and (8)

□