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THE INCONSISTENCY OF CERTAIN FORMAL LOGICS

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1. **Prospectus.** We discuss here some formal logics which are inconsistent in the sense that every formula in their notation is provable, irrespective of its meaning under the interpretation intended for the symbols. This results from the presence of a form of the Richard paradox, which we deduce, by utilizing representations of the logics within themselves, somewhat as follows:

A theory of positive integers $\mathbf{x}$ is constructed in the formal logic under consideration, such that the proposition $\mathbf{x}$ *is a positive integer* is expressible as a formula $N(\mathbf{x})$, and the proposition $\mathbf{Q}$ *is a positive integral function of positive integers* or *whenever* $N(x)$, *then* $N(\mathbf{Q}(x))$ as a formula $\mathbf{F}(\mathbf{Q})$. $\mathbf{F}(\mathbf{P})$ is proved for a given $\mathbf{P}$. Utilizing this theory, we select a representation of the formulas $\mathbf{A}$ of the logic by other formulas $\mathbf{a}$ of the same logic of such nature that, in general, the functions of the $\mathbf{a}$'s which correspond to metamathematical notions relating to the form of the $\mathbf{A}$'s are definable in the logic.[1] We require also a formula G which transforms each $\mathbf{a}$ into the logical product of the corresponding $\mathbf{A}$ and a provable formula $\mathbf{T}$, i.e. such that $G(\mathbf{a}) = \mathbf{A}$-*and*-$\mathbf{T}$ is provable. Then a formula $\mathfrak{H}$ is found which enumerates the representations of the provable formulas, i.e. such that for each $\mathbf{a}$ which represents a provable $\mathbf{A}$ there is at least one positive integer $\mathbf{x}$ such that $\mathfrak{H}(\mathbf{x}) = \mathbf{a}$ is provable, and conversely for each positive integer $\mathbf{x}$ there is an $\mathbf{a}$ which represents a provable $\mathbf{A}$ such that $\mathfrak{H}(\mathbf{x}) = \mathbf{a}$ is provable. A formula expressing *whenever* $N(x)$, *then* $G(\mathfrak{H}(x))$ is proved. A formula $\mathbf{L}$ is found such that $\mathbf{L}(\mathbf{a}) = \mathbf{a}$ is provable if $\mathbf{A}$ is of the form $\mathbf{F}(\mathbf{Q})$, and $\mathbf{L}(\mathbf{a}) = \mathbf{c}$, where $\mathbf{c}$ is the representation of $\mathbf{F}(\mathbf{P})$, is provable otherwise. Using G, a formula $\mathfrak{W}$ is found such that $\mathfrak{W}(\mathbf{a}) = \mathbf{Q}$ is provable if $\mathbf{a}$ is the representation of $\mathbf{F}(\mathbf{Q})$. Let $\mathbf{U}(x) = \mathfrak{W}(\mathbf{L}(\mathfrak{H}(x)))$. Then $\mathbf{U}$ enumerates the $\mathbf{Q}$'s such that $\mathbf{F}(\mathbf{Q})$ is provable. Using *whenever* $N(x)$, *then* $G(\mathfrak{H}(x))$, we prove *whenever* $N(x)$, *then* $\mathbf{F}(\mathbf{U}(x))$. Thence, letting $Q(x) = 1 + \{\mathbf{U}(x)\}(x)$, we prove $\mathbf{F}(Q)$. Hence Q is in the enumeration of $\mathbf{Q}$'s; but, formally, its x^{th} value exceeds the x^{th} value of the x^{th} $\mathbf{Q}$ by 1.[2]

In order that a G exist, the logic must permit great freedom in the expression of its formulas as values of functions. Logics have been proposed which have

[1] This type of representation was first used by K. Gödel in "*Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I,*" Monatshefte für Math. u. Physik, Vol. 38 (1931), pp. 173–198.

[2] Some of the considerations entering into this argument are discussed intuitively by A. Church in "*The Richard paradox,*" Am. Math. Monthly, Vol. 41 (1934), pp. 356–361.

this freedom, e.g. a combinatory logic of Curry[3] and a system of Church.[4] Both of these systems are inconsistent. Indeed, given the functional notation, together with the postulates governing it, of either,[5] a few additional properties suffice for the proof of inconsistency along the foregoing lines.

In the remaining sections we consider these results in greater detail.

2. Logics with the undefined terms $\{\ \}(\ )$, $\lambda[\ ]$, Π, Σ, &, **and variables.** We define *well-formed* formula, *free* and *bound* symbol, and *variable* as in Kleene 1934 §1, and use the abbreviations of Church and Kleene (except as otherwise provided), and the convention that letters in **heavy-type** denote well-formed expressions.

THEOREM A. *If in a given logic Church's Rules of Procedure* I–V[6] *are rules of procedure or valid methods of proof, and Church's Formal Postulates* 1, 3–11, 14–16[7] *are provable formulas, then in that logic every well-formed formula with no free variables is provable.*[8]

This theorem follows from Kleene 1935 thus:[9] Let $\mathbf{F} \rightarrow \lambda f \cdot N(x) \supset_x N(f(x))$ and $\mathbf{P} \rightarrow S$. Then $\mathbf{F}(\mathbf{P})$ is provable in C_1,[10] by conversion from 3.2. Choose $\mathbf{U}$

[3] H. B. Curry, "*Grundlagen der kombinatorischen Logik,*" Am. Jour. of Math., Vol. 52 (1930), pp. 509–536, 789–834; "*Some additions to the theory of combinators,*" *ibid.*, Vol. 54 (1932), pp. 551–558; "*The universal quantifier in combinatory logic,*" Annals of Math., Vol. 32 (1931), pp. 154–180; "*Apparent variables from the standpoint of combinatory logic,*" *ibid.*, Vol. 34 (1933), pp. 381–404; "*Some properties of equality and implication in combinatory logic,*" *ibid.*, Vol. 35 (1934), pp. 849–860. These papers will be cited by author and year.

The system of Curry which we are considering is that one whose postulates are the axioms and rules introduced in these five papers; our remarks do not apply, for example, to the system obtained by assuming only the axioms and rules given in the first four.

[4] A. Church, "*A set of postulates for the foundation of logic,*" Annals of Math., Vol. 33 (1932), pp. 346–366, and Vol. 34 (1933), pp. 839–864. The theory of these postulates is further developed by S. C. Kleene in "*Proof by cases in formal logic,*" Annals of Math., Vol. 35 (1934), pp. 529–544, and "*A theory of positive integers in formal logic,*" Am. Jour. of Math., Vol. 57 (1935), pp. 153–173, 219–244. We shall cite these papers by author and date.

[5] Or combinatory notation and postulates equivalent to the functional notation and postulates of the latter, as given by J. B. Rosser in "*A formal logic without variables,*" Annals of Math., Vol. 36 (1935), pp. 127–150, and Duke Math. Jour., Vol. 1 (1935), No. 3.

[6] By Church's Rules of Procedure I–V we shall mean the rules of Church 1932, p. 355–6 as revised in Kleene 1934 §1.

[7] Church 1932 p. 356, or 1933 p. 841.

[8] By *valid method of proof* we mean a rule the addition of which to the rules of procedure does not increase the class of provable formulas. (The terms *logic, rule of procedure* and *provable* may be understood as in Kleene 1934 p. 529.)

1, 3–11 can be replaced in this theorem by a set of formulas equivalent (when the rules of procedure are I–V, and 14–16 are axioms, and the interpretations of Kleene 1934 §1 are employed) to Church's Theorem *I* (1932, p. 358).

If the logic has axioms containing $\sim$ ("not") as a free symbol (such as Church's 17–27), then both $\mathbf{P}$ and $\sim \mathbf{P}$ are provable for every $\mathbf{P}$ having no free variables.

[9] We give this proof, although the theorem is a consequence of Thm. C, because the proof of the latter will be given only in outline and by comparison with this proof.

[10] C_1 denotes the logic whose rules of procedure and formal postulates are Church's I–V and 1, 3–11, 14–16, respectively. In accordance with the program stated at Kleene 1934 p. 530, "provable" at this point means "provable in C_1."

in accordance with 19XIII. By 19XIII(2), $N(n) \supset_n \mathbf{F}(\mathbf{U}(n))$, and, by conversion, (1) $N(n) \supset_n \cdot N(x) \supset_x N(\mathbf{U}(n, x))$. Thence, using 3.1, 5.2 and Thm. I, $N(n) \supset_n N(1 + \mathbf{U}(n, n))$, and by conversion $\mathbf{F}(Q)$, where $Q \rightarrow \lambda n \cdot 1 + \mathbf{U}(n, n)$. Thus $\mathbf{F}(Q)$ is provable in C_1. By 19XIII(1), there is a positive integer q such that (2) $\mathbf{U}(\mathrm{q})$ conv Q. $N(\mathrm{q})$ is provable from 3.1 by $q - 1$ successive applications of 3.2. Using (1), we infer (3) $N(\mathbf{U}(\mathrm{q}, \mathrm{q}))$. Now $1 = [1 + \mathbf{U}(\mathrm{q}, \mathrm{q})] - \mathbf{U}(\mathrm{q}, \mathrm{q})$ (11.2, 3.1, (3)), conv $[1 + Q(\mathrm{q})] - \mathbf{U}(\mathrm{q}, \mathrm{q})$ (by (2)), conv $[1 + \cdot 1 + \mathbf{U}(\mathrm{q}, \mathrm{q})] - \mathbf{U}(\mathrm{q}, \mathrm{q})$ (using the def. of Q), $= 2$ (conversion, 11.2, 3.1, 3.2, (3)). Hence, by §2, $1 = 2$. The conclusion follows by Kleene 1934 10I.

3. **Logics with the undefined terms $\{\ \}(\ )$, $\lambda [\ ]$, Π, &, and variables.** We now outline modifications of the preceding proof by which the use of Σ may be dispensed with.

Redefine E to be $\lambda x \cdot x = x$, and abbreviate $E(\mathrm{x}) \supset_{\mathrm{x}} \mathbf{M}$ to $`\mathrm{x} \cdot \mathbf{M}$.

THEOREM B. $\Pi(\lambda g \cdot g(\lambda f x \cdot f(x)), \lambda g \cdot g(\lambda f x \cdot f(f(x))))$ *is a provable formula in the logic whose rules of procedure and formal postulates are the following* (BI–BVII *and* B1–B19, *resp.*):

BI-BIII. *Church's Rules of Procedure* I–III.

BIV. *If* $\mathbf{F}(\mathbf{A})$, *then* $\Pi(\mathbf{F}, \mathbf{F})$.

BV. *If* $\mathbf{F}(\mathbf{A}) \cdot \Pi(\mathbf{F}, \mathbf{G})$, *then* $\mathbf{G}(\mathbf{A})$.

BVI. *If* $\mathbf{PQ}$, *then* $\mathbf{P}$.

BVII. *If* $\mathbf{P}$ *and* $\mathbf{Q}$, *then* $\mathbf{PQ}$.

B1. $p \supset_p \cdot pq \supset_q q$.

B2. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x g(x)] \supset_g \cdot [g(x) \supset_x h(x)] \supset_h \cdot f(x) \supset_x h(x)$.

B3. $`a \cdot `b \cdot g(a, b) \supset_g \cdot g(x, b) \supset_x E(x)E(b)$.

B4. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x E(x)E(b)] \supset_b \cdot [f(x) \supset_x g(x, b)] \supset_g \cdot f(x) \supset_x \Pi(g(x), g(x))$.

B5. $`a \cdot E(b(a)) \supset_b \cdot g(a, b(a)) \supset_g \cdot g(x, b(x)) \supset_x E(x)E(b(x))$.

B6. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x E(x)E(b(x))] \supset_b \cdot [f(x) \supset_x g(x, b(x))] \supset_g \cdot f(x) \supset_x \Pi(g(x), g(x))$.

B7. $`a \cdot f(a) \supset_f \cdot `b \cdot g(b) \supset_g \cdot [f(x) \supset_x g(b)\Pi(g, h(x))] \supset_h \cdot f(x) \supset_x h(x, b)$.

B8. $`a \cdot f(a) \supset_f \cdot q \supset_q \cdot f(x) \supset_x f(x)q$.

B9. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x E(b(x))\Pi(E, E)] \supset_b \cdot [f(x) \supset_x g(b(x))\Pi(g, g)] \supset_g \cdot [f(x) \supset_x g(b(x))\Pi(g, h)] \supset_h \cdot f(x) \supset_x h(b(x))$.

B10. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x g(x)] \supset_g \cdot [f(x) \supset_x h(x)] \supset_h \cdot f(x) \supset_x g(x)h(x)$.

B11. $`a \cdot E(b(a)) \supset_b \cdot g(b(a)) \supset_g \cdot g(b(x)) \supset_x \Pi(g, \lambda y \cdot E(x)E(y))$.

B12. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x E(b(x))\Pi(E, \lambda y \cdot E(x)E(y))] \supset_b \cdot [f(x) \supset_x g(b(x)) \Pi(g, \lambda y \cdot E(x)E(y))] \supset_g \cdot [f(x) \supset_x g(b(x))\Pi(g, h(x))] \supset_h \cdot f(x) \supset_x h(x, b(x))$.

B13. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x E(x)E(b)\Pi(\lambda y \cdot E(x)E(y), \lambda y \cdot E(x)E(y))] \supset_b \cdot [f(x) \supset_x g(x, b)\Pi(g(x), g(x))] \supset_g \cdot [f(x) \supset_x g(x, b)\Pi(g(x), h(x))] \supset_h \cdot f(x) \supset_x h(x, b)$.

B14. $`a \cdot E(b(a)) \supset_b \cdot g(a, b(a)) \supset_g \cdot g(x, b(x)) \supset_x \Pi(g(x), E)$.

B15. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x E(x)E(b(x))\Pi(\lambda y \cdot E(x)E(y), E)] \supset_b \cdot [f(x) \supset_x g(x, b(x))\Pi(g(x), E)] \supset_g \cdot [f(x) \supset_x g(x, b(x))\Pi(g(x), h)] \supset_h \cdot f(x) \supset_x h(b(x))$.

B16. $`a \cdot f(a) \supset_f \cdot [f(x) \supset_x E(x)E(b(x))\Pi(\lambda y \cdot E(x)E(y), \lambda y \cdot E(x)E(y))] \supset_b \cdot [f(x) \supset_x g(x, b(x))\Pi(g(x), g(x))] \supset_g \cdot [f(x) \supset_x g(x, b(x))\Pi(g(x), h(x))] \supset_h \cdot f(x) \supset_x h(x, b(x))$.

B17. $`a \cdot f(a) \supset_f \cdot q \supset_q \cdot f(x)q \supset_x f(x)$.

B18. $`a \cdot f(a) \supset_f \cdot g(a) \supset_g \cdot f(x)g(x) \supset_x f(x)$.

B19. $`a \cdot f(a) \supset_f \cdot p \supset_p \cdot f(x) \supset_x \cdot p \cdot f(x)$.

In proving this theorem, we shall use the following:

WEAK FORM OF CHURCH'S THEOREM I. *If the variable* $\mathbf{x}$ *does not occur in* $\mathbf{F}, \mathbf{G}$ *or* $\mathbf{M}$ *as a free variable, and if* $\mathbf{G}(\mathbf{x})$ *is provable as a consequence of* $\mathbf{F}(\mathbf{x})$ *and* $\mathbf{M}$, *then* $\mathbf{F}(\mathbf{x}) \supset_{\mathbf{x}} \mathbf{G}(\mathbf{x})$ *is provable as a consequence of* $\mathbf{F}(\mathbf{A})$ *and* $\mathbf{M}$.[11]

This is proved by induction on the number of applications of BIV–BVII in the proof of $\mathbf{G}(\mathbf{x})$ from $\mathbf{F}(\mathbf{x})$, $\mathbf{M}$ and the axioms (cf. the proof of Thm. I, Church 1932, pp. 358ff). For illustration we give one of the cases: Suppose the last application is the inference by BV of $\mathbf{h}(\mathbf{b})$ from $\mathbf{g}(\mathbf{b})\Pi(\mathbf{g}, \mathbf{h})$, where $\mathbf{x}$ is a free symbol of $\mathbf{b}$ and $\mathbf{h}$, but not of $\mathbf{g}$. Then, using the hypothesis of the induction, $\mathbf{F}(\mathbf{x}) \supset_{\mathbf{x}} \mathbf{g}(\mathbf{b}'(\mathbf{x}))\Pi(\mathbf{g}, \mathbf{h}'(\mathbf{x}))$ where $\mathbf{b}' \to \lambda\mathbf{x} \cdot \mathbf{b}$ and $\mathbf{h}' \to \lambda \mathbf{x} \cdot \mathbf{h}$. Using $\mathbf{F}(\mathbf{A})$, we obtain $\mathbf{g}(\mathbf{b}'(\mathbf{A}))$, and thence, by BIV, $\mathbf{g}(\mathbf{b}'(\mathbf{x})) \supset_{\mathbf{x}} \mathbf{g}(\mathbf{b}'(\mathbf{x}))$ or $\mathbf{g}(\mathbf{b}'(\mathbf{x})) \supset_{\mathbf{x}} \{\lambda \mathbf{xy} \cdot \mathbf{y}(\mathbf{b}'(\mathbf{x}))\}(\mathbf{x}, \mathbf{g})$. Using $E(\mathbf{A})$ and $E(\mathbf{g})$ (both obtained by use of BIV) and $\{\lambda \mathbf{xy} \cdot \mathbf{y}(\mathbf{b}'(\mathbf{x}))\}(\mathbf{A}, \mathbf{g})$ in B3, we obtain $\{\lambda \mathbf{xy} \cdot \mathbf{y}(\mathbf{b}'(\mathbf{x}))\}(\mathbf{x}, \mathbf{g}) \supset_{\mathbf{x}} E(\mathbf{x})E(\mathbf{g})$ or $\mathbf{g}(\mathbf{b}'(\mathbf{x})) \supset_{\mathbf{x}} E(\mathbf{x})E(\mathbf{g})$. The last two results with B4 yield $\mathbf{g}(\mathbf{b}'(\mathbf{x})) \supset_{\mathbf{x}} \Pi(\lambda \mathbf{y} \cdot \mathbf{y}(\mathbf{b}'(\mathbf{x})), \lambda \mathbf{y} \cdot \mathbf{y}(\mathbf{b}'(\mathbf{x})))$ or $\mathbf{g}(\mathbf{b}'(\mathbf{x})) \supset_{\mathbf{x}} E(\mathbf{b}'(\mathbf{x}))$. Now using B11, $E(\mathbf{b}'(\mathbf{x})) \supset_{\mathbf{x}} \Pi(E, \lambda \mathbf{y} \cdot E(\mathbf{x})E(\mathbf{y}))$; hence, by B2, $\mathbf{g}(\mathbf{b}'(\mathbf{x})) \supset_{\mathbf{x}} \Pi(E, \lambda \mathbf{y} \cdot E(\mathbf{x})E(\mathbf{y}))$; and, by

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[11] "$\mathbf{C}$ is provable as a consequence of $\mathbf{D}_1, \mathbf{D}_2, \cdots$" (or "$\mathbf{D}_1, \mathbf{D}_2, \cdots \vdash \mathbf{C}$") shall mean that $\mathbf{C}$ is derivable from $\mathbf{D}_1, \mathbf{D}_2, \cdots$ and the axioms of the logic under consideration at the time by means of its rules of procedure.

In practice, either the $\mathbf{M}$ is superfluous, or it stands for the logical product of previous "assumptions" made in preparation for further use of this and like theorems.

B10, $\mathbf{g}(\mathbf{b}'(\mathbf{x})) \supset_{\mathbf{x}} E(\mathbf{b}'(\mathbf{x}))\Pi(E, \lambda \mathbf{y} \cdot E(\mathbf{x})E(\mathbf{y}))$; also, by B18, $\mathbf{g}(\mathbf{b}'(\mathbf{x}))\Pi(\mathbf{g}, \mathbf{h}'(\mathbf{x})) \supset_{\mathbf{x}} \mathbf{g}(\mathbf{b}'(\mathbf{x}))$; and, by B2, $\mathbf{F}(\mathbf{x}) \supset_{\mathbf{x}} \mathbf{g}(\mathbf{b}'(\mathbf{x}))$; and hence, by B2, $\mathbf{F}(\mathbf{x}) \supset_{\mathbf{x}} E(\mathbf{b}'(\mathbf{x})) \Pi(E, \lambda \mathbf{y} \cdot E(\mathbf{x})E(\mathbf{y}))$. Similarly, $\mathbf{F}(\mathbf{x}) \supset_{\mathbf{x}} \mathbf{g}(\mathbf{b}'(\mathbf{x}))\Pi(\mathbf{g}, \lambda \mathbf{y} \cdot E(\mathbf{x})E(\mathbf{y}))$. Then, using B12, $\mathbf{F}(\mathbf{x}) \supset_{\mathbf{x}} \mathbf{h}'(\mathbf{x}, \mathbf{b}'(\mathbf{x}))$, and, by conversion, $\mathbf{F}(\mathbf{x}) \supset_{\mathbf{x}} \mathbf{G}(\mathbf{x})$.

We now parallel as closely as possible the theory given in Kleene 1935.

Whenever Kleene proved $\Sigma(\mathbf{F})$ and used Thm. I, he got the $\Sigma(\mathbf{F})$ by proving $\mathbf{F}(\mathbf{A})$ (except at certain points in the proof of 17.1), so the weak form of Thm. I can be used instead. However, without the Σ, we have no general method of stating implications on several variables. In some cases this causes no trouble. Thus we could just as well write 19.20 as $\mathrm{ad}(b)E(\mathfrak{G}(b)) \supset_b \cdot \mathrm{ad}(a)E(\mathfrak{G}(a, \mathfrak{G}(b))) \supset_a \cdot \mathfrak{G}(a, \mathfrak{G}(b)) = \mathfrak{G}(<a, b>)$. With this in mind, we parallel §§1–13.

However the material given in Kleene 1934 §8 (on which 1934 9I and 1934 10II depend) is not amenable to this sort of treatment, so we must find different proofs for the theorems proved by case arguments. Whenever "examples" can be found under both cases, we can use Thm. I and the definition of M instead. (More exactly, suppose that the case hypotheses $\mathbf{B} = 1$ and $\mathbf{B} = 2$ contain a free variable $\mathbf{x}$, that $\mathbf{R}$ is the logical product of all the other assumptions in which $\mathbf{x}$ occurs as a free variable, and that $\{\lambda \mathbf{x} \cdot \mathbf{R} \cdot \mathbf{B} = 1\}(\mathbf{X}_1)$ and $\{\lambda \mathbf{x} \cdot \mathbf{R} \cdot \mathbf{B} = 2\}(\mathbf{X}_2)$ can be proved. Then, by the case arguments and Thm. I, $[\mathbf{R} \cdot \mathbf{B} = 1] \supset_{\mathbf{x}} \mathbf{E}(\mathbf{x})\mathbf{C}$ and $[\mathbf{R} \cdot \mathbf{B} = 2] \supset_{\mathbf{x}} \mathbf{E}(\mathbf{x})\mathbf{C}$. By the def. of M and Thm. I, $M(\mathbf{w}) \supset_{\mathbf{w}} \cdot [\mathbf{R} \cdot \mathbf{B} = \mathbf{w}] \supset_{\mathbf{x}} E(\mathbf{x})\mathbf{C}$. $\mathbf{C}$ follows from $M(\mathbf{B})$, $\mathbf{R}$, and this formula.) Otherwise, we avoid the use of cases by more considerable changes in the proof, or modify the theorem in such a manner that examples become available.

To prove 14.14b, we use the lemma $M(a) \supset_a \cdot M(b) \supset_b \mathfrak{B}(4 - \cdot a + b, a, b)$, where $\mathfrak{B}(1)$ conv $\lambda ab \cdot M(a)M(b)$ and $\mathfrak{B}(2)$ conv $\lambda ab \cdot [a = 1]\, [b = 1]$.

The last clause of 15IV holds whenever a $\mathbf{T}$ can be found such that $\vdash \mathbf{T}(\mathbf{A})$ and $\mathbf{T}(\mathbf{X}) \vdash \mathbf{T}(\mathbf{F}(\mathbf{X}))$ for arbitrary $\mathbf{X}$. (For then $N(\mathbf{x}) \supset_{\mathbf{x}} \cdot \mathbf{T}(\mathbf{L}(\mathbf{x})) \cdot \mathbf{L}(\mathbf{S}(\mathbf{x})) = \mathbf{F}(\mathbf{L}(\mathbf{x}))$ is provable by induction.) The last clause of 15V holds whenever a $\mathbf{T}$ can be found such that $\vdash \mathbf{T}(\mathbf{A})$ and $N(\mathbf{Y})\mathbf{T}(\mathbf{X}) \vdash \mathbf{T}(\mathbf{F}(\mathbf{Y}, \mathbf{X}))$ for arbitrary $\mathbf{X}$ and $\mathbf{Y}$. (For then $\lambda \mathbf{x} \cdot \mathbf{T}(\mathbf{x}(1))$ is a "$\mathbf{T}$" for the application of 15IV in the proof of 15V.) 15IV and 15V as thus qualified suffice, since $\mathbf{T}$'s can be given for all the applications which we require, as follows: §16, $\mathfrak{S}$: $\lambda x \cdot N(x(I, 1))$. §17, $\mathfrak{B}$: $\lambda x \cdot [x(\lambda a \cdot I^a(1)) = \lambda m \cdot m(2, x(\lambda a \cdot I^a(1), \lambda pqr \cdot I^p(I^r(q))), 1)]\, [N(x(\lambda a \cdot I^a(1), \lambda pqr \cdot I^p(I^r(q))))]$; $\mathbf{L}$: N; $\mathbf{F}$: $\lambda \mathbf{x} \cdot \mathbf{T}(\mathbf{x}(1))$ (cf. 17I). §19, $\mathfrak{r}$, redefined so that $\mathfrak{r}(S(\mathbf{k}))$ conv $\lambda m \cdot m(\lambda pqr \cdot \mathfrak{r}(\mathbf{k}, q, I, I, \mathfrak{r}(\mathbf{k}, r, I, I, p)))$ $(k = 1, 2, \cdots)$: $\lambda x \cdot x(I, I, I, I) = I$; $\mathfrak{e}$ and $\mathfrak{e}'$: $\lambda x \cdot \mathrm{ad}(x([1]))$ (using $[1]_1 = [1]_2 = [1]$); $\mathfrak{D}$: $\lambda x \cdot x([1], [1]) = 2$; $\mathfrak{g}$: $\lambda x \cdot x([1]) = I$; $\mathfrak{i}$: ad. The remainder of §15 (after 15V) is not used.

We replace 17.1 by $[N(\xi) \supset_\xi \cdot r(\xi) > 1] \supset_r \cdot N(p) \supset_p \cdot [x < S(p) \supset_x \cdot y < S(r(x)) \supset_y E(f(x, y))] \supset_f \cdot [x < S(p) \supset_x \cdot y < S(r(x)) \supset_y t(f(x, y))] \supset_t \cdot z < S\left(\sum_{i=1}^{p} r(i)\right) \supset_z t(\mathfrak{Q}(f, r, z))$, and change 17.2 similarly. Then we have an example under Case 2 of (i), and prove $\left[N(\sigma) \cdot \mathfrak{C}_r(1, \sigma) \cdot \epsilon^{S(\sigma)}_{r(1)} = 1\right] \supset_\sigma$

$\mathfrak{C}_r(1, S(\sigma))$. This with the lemma $N(\tau) \supset_r \cdot [\phi(1) \cdot [N(\sigma)\phi(\sigma) \cdot \sigma < S(\tau)] \supset_\sigma \phi(S(\sigma))] \supset_\phi \phi(S(\tau))$ gives an example for Case 1. Under (ii), we define $\mathfrak{a}_r \rightarrow \lambda z \cdot S\left(\sum_{w=1}^{z} \epsilon\left(z, \sum_{i=1}^{w} r(i)\right)\right) - z$ and $\mathfrak{b}_r \rightarrow \lambda z \cdot \left\{\lambda a \cdot [z + r(a)] - \sum_{i=1}^{a} r(i)\right\} (\mathfrak{a}_r(z))$, and prove $N(p) \supset_p \cdot z < S\left(\sum_{i=1}^{p} r(i)\right) \supset_z \cdot \left\{\lambda ab \cdot a < S(p) \cdot b < S(r(a)) \cdot z = \left[\sum_{i=1}^{a} r(i) + b\right] - r(a)\right\}(\mathfrak{a}_r(z), \mathfrak{b}_r(z))$.

§18 is not used.

After reaching 19.21, certain changes are required to make the discussion apply to the new list of rules and axioms. We read $\Pi\&$ in place of $\Pi\Sigma\&$. The old $R_{39} - R_{614}$ we may regard as having been obtained by subdividing IV (*If* **F**(**P**), *then* Σ(**F**)) and V (*If* **F**(**P**) *and* Π(**F**, **G**), *then* **G**(**P**)) into $2^6 + 2^9$ rules R'_t ($t = 39, \cdots, 614$) by distinguishing cases according to the occurrence or non-occurrence of Π, Σ, and & in **F**, **P**, and **G** as free symbols, and then constructing the R_t from the R'_t according to a certain general pattern. We now subdivide BIV–BVII into $2^4 + 2^6 + 2^4 + 2^4$ rules R'_t ($t = 39, \cdots, 150$) with respect to the free occurrences of Π and &, and construct corresponding rules R_t in a precisely analogous manner. $\mathfrak{A}_1$–$\mathfrak{A}_{13}$ are replaced by combinations $\mathfrak{A}_1$–$\mathfrak{A}_{19}$ representative of B1–B19; etc.

Furthermore, we let C_γ denote the logic whose axioms are B1–B19 and whose rules of procedure are BI–BIII and a subset $R'_{\gamma_{39}}$–$R'_{\gamma_{p_\gamma}}$ of R'_{39}–R'_{150}; define $\gamma_v = v$ ($v = 1, \cdots, 38$); and replace C_1 by C_γ, R_t and $\mathfrak{R}_t$ ($t = 1, \cdots, 614$) by R_{γ_u} and $\mathfrak{R}_{\gamma_u}$ ($u = 1, \cdots, p_\gamma$), resp., m and n by m_γ and n_γ, resp., $\mathfrak{H}$ by $\mathfrak{H}_\gamma$, and **U** by $\mathbf{U}_\gamma$. Then, *if* we assume concerning γ that all the R'_{γ_w} ($w = 39, \cdots, p_\gamma$) are used in given proofs, we can finish paralleling §19, obtaining in conclusion 19XIII with C_1 and **U** replaced by C_γ and $\mathbf{U}_\gamma$, resp. (The assumption is used to provide examples for the cases in the proof of 19.23(u)).

Finally, we parallel the proof given in §2 of the present paper. First take $R'_{\gamma_{39}}$–$R'_{\gamma_{p_\gamma}}$ to be the cases of BIV–BVII which are used in the proof of 3.2. Then the assumption on γ under which we just proved 19XIII is satisfied, and also **F**(**P**) is provable in C_γ. Hence, using 19XIII(2), we obtain a proof of $\mathbf{F}(Q_\gamma)$, where $Q_\gamma \rightarrow \lambda n \cdot 1 + \mathbf{U}_\gamma(n, n)$. Now if this proof uses any of $\mathbf{R}'_{39}$–$\mathbf{R}'_{150}$ not in the list $R'_{\gamma_{39}}$–$R'_{\gamma_{p_\gamma}}$ (i.e. if it is not a proof in C_γ), we add them to the list, and repeat the argument with the new γ thus obtained; and so on. Eventually we obtain a γ such that $\mathbf{F}(Q_\gamma)$ is proved in C_γ. Then using 19XIII(1), we can complete the proof of $1 = 2$. By conversion, $\Pi(\lambda g \cdot g(\lambda fx \cdot f(x)), \lambda g \cdot g(\lambda fx \cdot f(f(x))))$.

THEOREM C. *If in a given logic* CI–CVI *of the list which follows are rules of procedure or valid methods of proof, and* C1 *and* C2 *are provable formulas, then in that logic every well-formed formula with no free variables is provable*:

CI–CIII. *Church's Rules of Procedure* I–III.

CIV. *If* **PQ**, *then* **P**.

CV. *If* $\mathbf{F}(\mathbf{A})$ *and* $\mathbf{F}(\mathbf{x}) \supset_{\mathbf{x}} \mathbf{G}(\mathbf{x})$, *where* $\mathbf{x}$ *is a variable not occurring in* $\mathbf{F}$ *or* $\mathbf{G}$, *then* $\mathbf{G}(\mathbf{A})$.

CVI. *The weak form of Church's Theorem* I.

C1. $p \supset_p \cdot q \supset_q pq$.

C2. $p \supset_p \cdot pq \supset_q q$.

Proof. Let $\Pi' \rightarrow \lambda fg \cdot f(x) \supset_x g(x)$. Then if we replace Π by Π' in the logic of Thm. B, its rules of procedure become valid methods of proof, and its formal postulates become provable formulas in the given logic. Hence

$$\Pi'(\lambda g \cdot g(\lambda fx \cdot f(x)), \lambda g \cdot g(\lambda fx \cdot f(f(x))))$$

is provable in the same. By conversion, $1 = 2$. The conclusion follows by Kleene 1934 10I (which holds good for the given system).

4. **Combinatory logics.** Rosser has shown that the functional notation and rules of Church can be cast in combinatorial form. Consequently Thms. A-C have combinatorial equivalents.[12]

We turn now to the above-mentioned system of Curry.

THEOREM D. *If in a given logic Curry's Rules E-P*[13] *are rules of procedure or valid methods of proof, and Curry's Axioms* Q-I_2, (ΠB)–(ΠP), Π_0, (PB)–(PK)[14] *are provable formulas, then in that logic every entity*[15] *is provable.*

Proof. Define $\lambda y[Y]$ to be $[y]Y$, $\{F\}(A)$ to be (FA), Π to be $[f, g](x)(fx \supset gx)$, and & to be $[p, q]\,(x)\,((f)\,(fK \supset (f(CI) \supset fx)) \supset xp(Kq))$.[16] Then it can be shown that BI–BVII hold and B1–B19 are provable. Hence, by Thm. B, $\vdash \Pi(\lambda g \cdot g(\lambda fx \cdot f(x)), \lambda g \cdot g(\lambda fx \cdot f(f(x))))$. $\vdash (g)\ (gI \supset g(WB))$ follows combinatorially. Hence, letting T be any entity such that $\vdash T$ and F be any entity, $\vdash ([x]xCKFT)I \supset ([x]xCKFT)(WB)$. $\vdash ([x]xCKFT)I$ follows combinatorially from $\vdash T$. Hence $\vdash ([x]xCKFT)(WB)$, from which $\vdash F$ follows combinatorially.

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[12] To obtain such equivalents, we need only to utilize the definitions of I, J and *combination*, and Thm. 6V, of Kleene 1934 §6, and the property of the thirty-eight rules R_1–R_{38} of Rosser, *loc. cit.*, Section H that *if* $\mathbf{A}$ *and* $\mathbf{B}$ *are combinations, and* $\mathbf{A}$ *conv* $\mathbf{B}$, *then* $\mathbf{A}$ *is derivable from* $\mathbf{B}$ *by* R_1–R_{38}.

[13] Curry 1930 p. 522.

[14] Curry 1930 p. 521, 1931 p. 169, 1933 p. 399, 1934 p. 850.

[15] We use *entity* here in Curry's sense (cf. 1930 I C and 1931 p. 157). In particular, every *combination* of Curry's B, C, W, K, Q, Π, P, Λ is an entity.

[16] Here Y, F and A represent combinations, and y represents a variable occurring in Y.

To facilitate the statement of the relation between the present system and that of Thm. B, we suppose Curry's Π replaced by some other symbol, and (without invalidating Thm. B) restrict the bound symbols of well-formed expressions to variables in Curry's sense.