Abstract-value Slicing

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Abstract

In our previous work [SYY03] we have shown that it is possible to automatically construct Hoare proofs of programs by using the abstract interpreters to approximate the programs’ invariants. The next question is: would the proposed technique be effective? Because the automatic proof construction is motivated by the Proof Carrying Code framework, we are keen to minimize the generated proofs.

We observed that when an abstract interpreter is used to verify a specific safety property, it often computes invariants that are not needed for the verification. The reason is that the abstract interpreter does not know what the safety property is, so it simply tries to find as strong invariants as possible. These useless invariants result in an unnecessarily big proof. Unless the proof-construction phase is notified which invariants are useless, it cannot help but proving all the computed invariants.

In this paper, we present an algorithm that slices out useless invariants from the abstract interpretation results. This algorithm works as a post-processor to an abstract interpreter in the whole proof-construction process, and notifies to the next proof-construction phase which invariants it does not have to prove. In our experiment with Miné’s abstract interpretation, the algorithm identified 63% – 84% of the computed invariants as useless, and resulted in 56% – 88% reduction in the size of constructed proofs.

1 Introduction

The Proof-Carrying Code (PCC) technologies [NS02, NR01, App01, HST+02] have been a convincing approach for certifying the safety of mobile code, yet how to achieve the code's safety proof is still open for alternatives. The existing proof construction process is either not fully automatic, assuming that the program invariants should be provided by the programmer [Nec97, NL97, NR01], or limited to a class of properties that are automatically inferable by the current type system technologies [HST+02, AF00, MWCG98].

In our previous work [SYY03], we have shown that by combining static analysis and program logic, we can automatically construct proofs for a wider class of program properties. We use Hoare logic [Hoa69] for representing the proofs of program properties, and the abstract interpretation [CC77, Cou99] for computing the program properties. In this combination, an abstract interpretation automatically estimates program properties (approximate invariants) of our interests, and our proof-construction method constructs a Hoare proof for those approximate invariants.
This paper is motivated by one problem with our proof-construction method. When a Hoare proof of a property is constructed by our method, the substantial parts of the proof are often useless; they do not contribute to the verification of the property. As a result, the constructed Hoare proof is unnecessarily big.

The main reason for this problem is that the abstract interpreter computes approximate invariants stronger than what are needed for verification. When we use an abstract interpreter to prove a specific safety property of a program, the interpreter does not know what the property is; it simply tries to discover as strong invariant as possible. As a result, it usually finds some (approximate) program invariants that are not needed to prove the safety property. Note that this existence of useless invariants becomes a bottleneck for all the efforts to reduce the size of a proof; if the proof construction does not know which invariants are of no use, it cannot help but proving all the invariants that the abstract interpreter found.

To see this problem more clearly, let’s consider the following insertion sort program annotated with the results of Miné’s abstract interpretation [Min01a]:

```
sort(int n, int A[1..n])
{
  int i,j,pivot;
  // true
  i:=2; j:=0;
  // inv: (2\leq i) \land (0\leq j\leq i+2)
  while (i<n) {
    // (2\leq i) \land (0\leq j\leq i+2)
    pivot:=A[i]; j:=i-1;
    // inv: (2\leq i) \land (0\leq j\leq i-1) \land (2\leq n) \land (j\leq i-1)
    while (j=1 and A[j]>pivot) {
      // (2\leq i) \land (1\leq j\leq n-1) \land (2\leq n) \land (j\leq i-1)
      A[j+1]:=A[j]; j:=j-1;
    } // (2\leq i) \land (0\leq j\leq n-1) \land (2\leq n) \land (j\leq i-1) --- *
    A[j+1]:=pivot; i:=i+1;
  }
  // (2\leq i) \land (0\leq j\leq n-1)
}
```

This program takes an array A and the size n of the array as an input, and sorts the array. Suppose that we ran the abstract interpreter in order to verify the absence of array index errors in the program. The annotations in the program prove this safety property, but they also contain unnecessary information. For instance, \( i\leq n \) in the annotation marked by * neither shows that the following array access \( A[j+1] \) is within bounds, nor is used to imply the loop invariant \( (2\leq i) \land (0\leq j\leq i+2) \). Thus, it can be eliminated without breaking the proof. In fact, half of the annotations in the program are not needed. If all such useless invariants are eliminated, the program becomes:

```
sort(int n, int A[1..n])
{
  int i,j,pivot;
  // true
  i:=2; j:=0;
  // inv: 2\leq i
  while (i<n) {
    // 2\leq i
    pivot:=A[i]; j:=i-1;
    // inv: (2\leq i) \land (0\leq j\leq n-1)
    while (j=1 and A[j]>pivot) {
      // (2\leq i) \land (1\leq j\leq n-1)
      A[j+1]:=A[j]; j:=j-1;
    } // (2\leq i) \land (0\leq j\leq n-1)
  }
}
```

\(^1\)Here we assume that in “\( B_1 \) and \( B_2 \)”, \( B_2 \) is evaluated only when \( B_1 \) is true.
A[j+1]:=pivot; i:=i+1;

In this paper, we present an algorithm that slices out such useless invariants from the results of an abstract interpreter. Our algorithm works as a post-processor to the abstract interpreter. Given an annotated program and a property of interests, the algorithm approximates all the annotations further, until all the information in each annotation contributes to the verification of the property.

The main merit of our solution is that it does not require any changes from an abstract interpreter. An alternative approach for this problem of useless invariant is to modify an abstract interpreter, so that the interpreter becomes the goal-oriented backward one; the modified interpreter computes an under-approximation of “safe” initial states, the states from which a program always achieves the given goal. However, such an approach needs a complete redesign of the analysis, because most forward abstract interpreters are not straightforward to have their backward versions, and some of them cannot be used to model under-approximation. For example, Miné’s abstract domain $\mathcal{M}$ [Min01a] does not have an “under-approximation” map $\delta$: there are no monotone functions $\delta: \mathcal{M} \rightarrow \mathcal{P}(\text{States})$ such that

$$\forall S \in \mathcal{P}(\text{States}), \forall m \in \mathcal{M}. m \subseteq \mathcal{M} \delta(S) \iff \gamma(m) \subseteq S.$$

where $\gamma$ is the concretization map for $\mathcal{M}$. The reason is that the existence of such an under-approximation implies that $\gamma$ preserves all joins, but there are some joins that $\gamma$ does not preserve.

In this paper, we first explain our solution with Miné’s abstract domain [Min01a]. In Section 2 and 3, we review his abstract interpretation, and propose an abstract-value slicing algorithm that eliminates useless invariants from the abstract interpretation results. In Section 4, we explain the experimental results from our prototype implementation. The results show that our algorithm sliced out the 63%–84% of the computed invariants, so that it induced from 56% to 88% reduction in the size of the constructed proofs. Then, we propose a general framework for abstract-value slicing in Section 5, and conclude the paper in Section 6.

## 2 Miné’s Abstract Interpreter

Miné’s abstract interpreter [Min01a] estimates, at each program point, the upper and lower bounds of the “distance” between two program variables (i.e., $x_i - x_j$). In this section, we review this abstract interpreter.

### 2.1 Programming Language

We assume that the abstract interpreter is given programs in a simple imperative language with integer variables. The syntax of the language is given below:

**Boolean Expressions**

$$B ::= x-y \leq c \ | \ * \ | \ B \land B \ | \ B \lor B$$

**Commands**

$$C ::= x:=y+c \ | \ \text{skip} \ | \ C;C \ | \ \text{if} B \ \text{then} C \ \text{else} C \ | \ \text{while} B \ \text{do} C$$

Note that the language allows only very limited forms of assignments and boolean expressions. The right-hand side of an assignment should be the $c$ increment of a variable $y$; and the atomic boolean expression should have the form of $x - y \leq c$. We consider only these special forms of assignments and boolean expressions in order to focus on the key aspects of Miné’s abstract interpreter; his interpreter can handle such assignments and boolean expressions well, but for other more general forms of assignments and boolean expressions, it merely uses the standard interval domain [CC77].
Another feature of the language is that it includes a boolean expression * that nondeterministically evaluates true or false. This * expression normally fills in the place where an expression has been sliced out before the analysis is applied.

We frequently use a syntactic sugar \( \neg B \). It is defined inductively using the usual de-Morgan laws\(^2\) and the following equivalence:

\[
\neg(x - y \leq c) \Leftrightarrow y - x \leq -c - 1.
\]

## 2.2 Abstract Domain

The abstract domain for Miné’s abstract interpreter consists of difference-bound matrices (in short, DBMs). Let \( N \) be the number of the program variables in a given program, and \( x_1, \ldots, x_N \) an enumeration of all those variables. A DBM \( m \) for this program is an \((N+1) \times (N+1)\) matrix with integer values, \( -\infty \) or \( \infty \). Intuitively, each \( m_{ij} \) entry denotes the upper bound of \( x_j - x_i \) (that is, \( x_j - x_i \leq m_{ij} \)). A DBM \( m \) means the conjunction of all these constraints \( m_{ij} \) and \( x_0 = 0 \); because of this additional constraint about \( x_0 \), the range of a variable can be expressed by a DBM. For example, a DBM

\[
\begin{array}{ccc}
x_0 & x_1 & x_2 \\
x_0 & \infty & \infty & \infty \\
x_1 & \infty & \infty & -1 \\
x_2 & 0 & \infty & \infty
\end{array}
\]

means \( x_0 - x_2 \leq 0 \land x_2 - x_1 \leq -1 \land x_0 = 0 \), which is equivalent to \( -x_2 \leq 0 \land x_2 - x_1 \leq -1 \).

Let \( \text{States} \) be the set of states (i.e., \( \text{States} = \text{Ints}^{\{x_1, \ldots, x_N\}} \)). Formally, the abstract domain is defined by the lattice \( M = (M, \sqcap, \sqcup, \sqsubseteq, \sqcap, \sqcup) \) of DBMs, concretization map \( \gamma: M \rightarrow \mathcal{P}(\text{States}) \), and abstraction map \( \alpha: \mathcal{P}(\text{States}) \rightarrow M \):

\[
\begin{align*}
M & \triangleq \{ m \mid m \text{ is a DBM}\} \quad m \sqsubseteq m' \iff \forall i, j. m_{ij} \leq m'_{ij} \\
\sqcap_{ij} & \triangleq \infty \quad \sqcup_{ij} \triangleq -\infty \\
[m \sqcup m']_{ij} & \triangleq \max(m_{ij}, m'_{ij}) \quad [m \sqcap m']_{ij} \triangleq \min(m_{ij}, m'_{ij}) \\
\gamma(m) & \triangleq \{ s \in \text{States} \mid \forall i, j. s(x_0 - 0)(x_i) - s(x_0 - 0)(x_j) \leq m_{ij} \} \\
\alpha(S)_{ij} & \triangleq \max(\{ s(x_0 - 0)(x_i) - s(x_0 - 0)(x_j) \mid s \in S \})
\end{align*}
\]

In the definition of \( \gamma \) and \( \alpha \), we used \( s(x_0 - 0) \), the extension of state \( s \) with an additional component for \( x_0 \):

\[
s(x_0 - 0)(x_i) \trianglerighteq 0 \text{ if } (i = 0) \text{ then } 0 \text{ else } s(x_i).
\]

One special feature of this abstract domain is that it has a lower closure operator \( \neg^*: M \rightarrow M \). Miné’s domain is slightly different from other common abstract domains in that the concretization is not injective: two different DBMs might mean the same set of states. Moreover, among such DBMs with the same meaning, there is the “best representation” \( m \): for all \( m \), if \( \gamma(n) = \gamma(m) \), then \( n \sqsubseteq m \). The closure operator \( \neg^* \) transforms each DBM \( m \) to this best \( n \). That is, \( m^* \sqsubseteq m \), \( \gamma(m^*) = \gamma(m) \), and \( m^* \) is the smallest such DBM. More explicitly, the closure operator works as follows. Given a DBM \( m \), it first checks whether \( m \) has a “negative cycle”: a nonempty sequence \( i_1 \ldots i_n \) such that \( i_1 = i_n, 0 \leq i_1 \ldots i_n \leq N \), and \( \sum_{k=1}^{n-1} m_{i_k,i_{k+1}} < 0 \). If so, the closure \( m^* \) is \( \bot \). Otherwise, each \( (i, j) \) entry is updated by the “length” of a shortest path from \( i \) to \( j \) in \( m \): when \( i = j \), the \( (i, j) \) entry of \( m^* \) is 0; and when \( i \neq j \), the \( (i, j) \) entry is

\[
\min \left\{ \sum_{k=1}^{n-1} m_{i_k, i_{k+1}} \mid 0 \leq i_1 \ldots i_n \leq N \land i_1 = i \land i_n = j \right\}.
\]

\(^2\)\(\neg(B_1 \land B_2) \iff (\neg B_1 \lor \neg B_2) \text{ and } (\neg B_1 \lor B_2) \iff (\neg B_1 \land \neg B_2).\)
For example,
\[
\begin{pmatrix}
 x_0 & x_1 & x_2 \\
 x_0 & \infty & \infty \\
 x_1 & \infty & -1 \\
 x_2 & \infty & \infty \\
\end{pmatrix}^* = \begin{pmatrix}
 x_0 & x_1 & x_2 \\
 x_0 & 0 & \infty \\
 x_1 & -1 & 0 \quad -1 \\
 x_2 & 0 & \infty \quad 0
\end{pmatrix}.
\]

Note that in the closed DBM, all the diagonal entries became 0 by definition \((i = j)\) case; and the value of the \(x_1x_2\) entry has changed to \(-1\), because there is a path \(x_1x_2x_0\) from \(x_1\) to \(x_0\) whose “length” is \(-1\).

In the abstract interpreter, we use the closure operator \(-^*\) in order to improve the accuracy of the analysis.

## 2.3 Abstract Interpreter

Given a program, the abstract interpreter computes an invariant DBM at each program point, and outputs an annotated program \(A\) of the following form:
\[
A ::= \text{\lbrack}m\text{\rbrack}R[m] | A; A | \text{if } B \text{ then } A \text{ else } A | \text{inv } m \text{ while } B \text{ do } A
\]

The computed DBM annotations in \(A\) satisfy a set of constraints that imply the correctness of the annotations: if the constraints hold, then the annotations in \(A\) approximate program invariants. Figure 1 shows the algorithm \(C\) that generates such constraints. We note the two cases: one for the assignment of the form \(x_i:=x_j+c\) with \(i \neq j\), and the other for a while loop.

Given an assignment of the form \([m]x_i:=x_j+c[m]\), the algorithm \(C\) first abstractly executes the assignment \(x_i:=x_j+c\): it first transforms \(m\) so that the DBM has the best representation \(m^*\); then, it eliminates all the information in \(m^*\) involving the old value of \(x_i\); finally, it adds a fact that \(x_i-x_j\leq c\) and \(x_j-x_i \leq -c\). Once finishing this abstract execution, \(C\) outputs a constraint which means that the result of the abstract execution should be approximated by \(n\). For a while loop
\[
[m][\text{inv } i \text{ while } B \text{ do } (\text{[m] } R[m]))[n],
\]
the algorithm produces the four constraints besides those for the loop body:
\[
m^* \subseteq i, \ (n_1)^* \subseteq i, \ [B][\text{case } B][m_1], \text{ and } [\neg B][\text{case } \neg B] \subseteq n.
\]

The first two constraints express that \(i\) abstracts the loop invariant. The DBM \(i\) should approximate all the initial states (i.e., \(m^* \subseteq i\)) as well as all the states at the end of the loop (i.e., \((n_1)^* \subseteq i\)). The other constraints are about the states that pass or fail the test \(B\). The third constraint prunes some states in \(\gamma(i)\) that do not pass \(B\), and asks that \(m_1\) should approximate all the remaining states in \(\gamma(i)\) (i.e., \([B][\text{case } B] \subseteq m_1\)). Note that this constraint implies that \(\gamma(m_1)\) includes all the states in \(\gamma(i)\) that pass the test \(B\). Similarly, the fourth constraint implies that \(n\) approximates all the states in \(\gamma(i)\) that fail the test \(B\).

Figure 2.(a) shows an annotated program \(A\) whose DBMs satisfy the constraints generated by \(C\). Among those generated constraints, we note the following two for the loop invariant DBM:
\[
\begin{pmatrix}
 x_0 & x_1 & x_2 \\
 x_0 & \infty & 0 \\
 x_1 & \infty & \infty \\
 x_2 & \infty & \infty
\end{pmatrix}^* \subseteq \begin{pmatrix}
 x_0 & x_1 & x_2 \\
 x_0 & \infty & \infty \\
 x_1 & \infty & -1 \\
 x_2 & 0 & \infty
\end{pmatrix} \subseteq \begin{pmatrix}
 x_0 & x_1 & x_2 \\
 x_0 & \infty & \infty \\
 x_1 & 0 & \infty \\
 x_2 & \infty & \infty
\end{pmatrix}
\]

The same DBM on the right-hand side of both constraints is a (approximate) loop invariant of the program, and it means that \(-x_1 \leq 0\). The first constraint ensures that the invariant holds before the loop gets executed, and the second that it holds right after the loop body.
about “crucial DBM entries”: the entries of the DBMs in the invariants is necessary for verification. When the abstract interpreter in Section 2 is used for verification, it usually computes stronger invariants than what are needed. The abstract interpreter does not know what properties we want to verify, so it simply tries to find as stronger invariants as possible. In this section, we present the abstract-value slicer that weakens the computed invariants until all the information in the invariants is necessary for verification.

In this paper, we will not specify a particular strategy to obtain annotations satisfying the constraints. It is because the correctness of the abstract-value slicer in the paper does not depend on such a strategy; it only needs that the computed DBM annotations satisfy the constraints. It is because the correctness of the abstract-value slicer in the paper does not depend on such a strategy; it only needs that the computed DBM annotations satisfy the constraints. If a reader is interested in a particular strategy, we refer to Miné’s original paper [Min01a], where he computes the correct annotations with specific widening and narrowing operators.

### 3 Abstract-value Slicer

When the abstract interpreter in Section 2 is used for verification, it usually computes stronger invariants than what are needed. The abstract interpreter does not know what properties we want to verify, so it simply tries to find as stronger invariants as possible. In this section, we present the abstract-value slicer that weakens the computed invariants until all the information in the invariants is necessary for verification.

The abstract-value slicer takes the output $A$ of the abstract interpreter, and information about “crucial DBM entries”: the entries of the DBMs in $A$ that are used for verification. More

![Pruning Algorithm](image)

Both of these constraints hold; whenever an entry of the DBM on the left is bigger than the corresponding entry of the other DBM, the closure operator changes it to a smaller value. For instance, in the second constraint, the $x_1x_2$ entry of the DBM on the left is $\infty$, which is bigger than the corresponding entry of the DBM on the right. The closure $-^*$ fixes this “problem”, by computing a shortest path $x_1x_2x_0$, and replacing the value $\infty$ of the $x_1x_0$ entry by $-1$, the “distance” of this path.

In this paper, we will not specify a particular strategy to obtain annotations satisfying the constraints. It is because the correctness of the abstract-value slicer in the paper does not depend on such a strategy; it only needs that the computed DBM annotations satisfy the constraints. If a reader is interested in a particular strategy, we refer to Miné’s original paper [Min01a], where he computes the correct annotations with specific widening and narrowing operators.
precisely, the second input component is a program \( T \) annotated with index sets:

\[
T := [I]S[I] \\
S := x_i := x_j + c | T:T | if B then T else T | [inv]while B do T
\]

Here \( I \) denotes a set of DBM indices. We assume that the two components \( A \) and \( T \) in an input are about the same program, so that their underlying programs are identical. Each index set \( I \) in \( T \) indicates that among all the entries in the corresponding DBM \( m \), only those in \( I \) are directly used to verify a safety property. For example, suppose that we have used the annotated program \( A \) in Figure 2.(a) to prove that \( x_1 \geq 0 \) holds at the end of the program. In this case, only the \( x_1x_0 \) entry of the last DBM annotation is crucial, so we use the following \( T_{cr} \) as an input to the slicer:

\[
[\emptyset] ( [inv : \emptyset] \text{while } (x_1 - x_2 \leq 0) \text{ do } \emptyset | x_1 := x_2 + 1[\emptyset] ) \{ \{1,0\}\}
\]

The result of the abstract-value slicer is an index-set annotation \( T' \) that records which DBM entries are used to compute the “crucial DBM entries.” Suppose that the slicer is given a DBM annotation \( A \) and an index-set annotation \( T \). The slicer returns an index-set annotation \( T' \) that satisfies the following three properties:

1. The underlying program \( T' \) is identical to the underlying program \( T \).
2. Every index set $I$ in $T$ is included in the corresponding index set $I'$ in $T'$.

3. For an index set $I$ and a DBM $m$, let $\text{prj}(m, I)$ be the projection of $m$ by $I$:
   \[ \text{prj}(m, I)_{ij} = \begin{cases} m_{ij} & \text{if } (i, j) \in I \\ \infty & \text{otherwise} \end{cases} \]

   If we project every DBM $m$ in $A$ by the corresponding index set $I'$ in $T'$, then the annotations in the resulting program satisfy the constraints generated by $C$.

The third property ensures that we can safely “eliminate” all the matrix entries not in $T'$, and the second property means that even when we eliminate such entries, the remaining ones are still strong enough to obtain all the “crucial” information. For instance, suppose that we ran the slicer with the annotated program $A$ in Figure 2.(a) and the index-set annotation $T_{ex}$ in the previous paragraph. Then, the slicer computes, at each program point, which DBM entries are used to prove that $x_1 \geq 0$ holds at the end of the program. Figure 2.(b) shows the computed index-set annotation $T'$, and the projection of each DBM in $A$ by the corresponding index set in $T'$. Note that these results identify three invariants in the original annotation $A$ as “useless”: $x_1 \leq 0$ in the first DBM, $x_1 - x_2 \leq 1$ in the DBM right after $x_2 := x_2 + 1$, and $x_2 - x_1 \leq -1$ in the last DBM. The first $x_1 \leq 0$ is not necessary, because the other invariant $-x_1 \leq 0$ in the first DBM is strong enough to imply the approximate loop invariant $-x_1 \leq 0$. By the same reason, the second is not needed either. Note that the arguments for these two invariants do not depend on the input $T_{ex}$. On the other hand, the reason that we can eliminate $x_2 - x_1 \leq -1$ comes from $T_{ex}$. Since $x_2 - x_1 \leq -1$ is the property of the final state, but it does not “belong” to $T_{ex}$, we can eliminate $x_2 - x_1 \leq -1$.

Figure 3 shows the definition of the abstract-value slicer $\text{slicer}$ in an imperative style. The second input component to $\text{slicer}$ is a program annotated with variables for index sets. The algorithm $\text{slicer}$ updates these index-set variables directly, so that when $\text{slicer}$ terminates, the variables contain which DBM entries are used to compute crucial DBM entries (i.e., the initial values of the variables).

Algorithm $\text{slicer}$ uses two subroutines $\text{clsr}$ and $\text{bslice}$. For an index set $I$ and a pair of DBM matrices $m$ and $n$ such that $m^* \subseteq n$, the first subroutine $\text{clsr}(m, n, I)$ discovers which entries of $m$ are needed to get the $I$ entries of $n$ by the closure and the order $\subseteq$. Let $(i, j)$ be an index in $I$. If $m_{ij} \leq n_{ij}$, the $m_{ij}$ entry of $m$ (i.e., $x_j - x_i \leq m_{ij}$) alone implies the $n_{ij}$ entry of $n$ (i.e., $x_j - x_i \leq n_{ij}$). The subroutine, therefore, simply puts $(i, j)$ to the resulting index set. On the other hand, if $m_{ij} > n_{ij}$, the $(i, j)$ entry of $m$ is not strong enough; in order to get the $(i, j)$ entry of $n$, we need to chain several entries in a “shortest path” from $i$ to $j$ in $n$. Thus, in this case, the subroutine calls $\text{mPath}$ to obtain a shortest path from $i$ to $j$, and adds all the entries in the path to the result index set.

The other subroutine $\text{bslice}$ is about the case pruning. Given a boolean expression $B$, DBMs $m, n$, and an index set $I$ such that $[B]m \subseteq n$, it computes which entries of $m$ are needed to get the $I$ entries of $n$. For instance, when $B$ is $x_i - x_j \leq c$, subroutine $\text{bslice}$ checks whether $B$ “implies” the $x_j x_i$ entry of $n$ (i.e., $x_i - x_j \leq c \Rightarrow x_i - x_j \leq n_{ji}$). If so, $\text{bslice}$ removes $(j, i)$ from $I$. Otherwise, it keeps the $(j, i)$ entry in $I$. Note that this case exploits the value of $n_{ji}$, which is available only after the abstract interpreter computes approximate invariants.

The definition of slice is given inductively on the structure of the arguments. We note the case for the assignment $x_i := x_j + c$ with $(i \neq j)$. Given $[m][x_i := x_j + c][n]$ and $[X][x_i := x_j + c][Y]$, algorithm $\text{slicer}$ backtracks the abstract execution of the assignment. Recall that the abstract execution of $x_i := x_j + c$ first replaces $m$ by a better representation $m^*$, and then modifies $(i, -)$ and $(-, i)$ entries of $m^*$: it eliminates $(i, -)$ and $(-, i)$ entries, and updates $(j, i)$ to $c$, and $(i, j)$ to $-c$. The first step of the slicer $\text{slicer}$ is to roll-back the last updates. For all $k$, the slicer $\text{slicer}$ removes $(k, i)$ and $(i, k)$ from $Y$, and replaces the values of $(k, i)$ and $(i, k)$ entries of $n$ by $\infty$. This backtracking of the updates reflects that the abstract interpreter computed $(k, i)$ and $(i, k)$ entries without reading any entries of the initial DBM $m$. The next step of the slicer is
clsr(m, n, I) = if (m^* = \bot) then \emptyset else \bigcup_{(i,j) \in I} \left( \left\{ (i,j) \right\} \right) \left( \left\{ \text{if } m_{ij} \leq n_{ij} \text{ then } \{ (i, j) \} \text{ else } mPath(m, (i, j)) \right\} \right)

bslice(x_i - x_j \leq c, m, n, I) = \left\{ \begin{array}{ll}
I - \{(j,i)\} & \text{if } (n_{ji} \geq c) \\
I & \text{otherwise}
\end{array} \right.

bslice(B_1 \lor B_2, m, n, I) = bslice(B_1, m, n, I) \cup bslice(B_2, m, n, I)

bslice(B_1 \land B_2, m, n, I) = \text{let } I_1 = \{(i,j) \in I \mid (\lceil B_1 \rceil)_{m_{ij}} \leq n_{ij}\} \\
I_2 = \{(i,j) \in I \mid (\lceil B_2 \rceil)_{m_{ij}} \leq n_{ij}\}
\text{in } bslice(B_1, m, n \cup [B_1]_m, I_1) \\
\cup bslice(B_2, m, n \cup [B_2]_m, I_2)

\text{slice}([m]skip[n], [X]skip[Y])
\triangleq X \assign X \cup Y

\text{slice}([m]x_i := x_j + c[n], [X]x_i := x_j + c[Y])
\triangleq \text{if } (i = j) \\
& \text{ then } X \assign X \cup Y \\
& \text{ else } X \assign X \cup \text{clsr}\left(m, (n_{ij} \rightarrow \infty, (k, i) \rightarrow \infty)_{0 \leq k \leq N}, Y - \{(i,k), (k,i) \mid 0 \leq k \leq N}\right)

\text{slice}([m][m_1]R_1[m_1]; [m_2]R_2[n_2])\{n\}, [X][[X_1]S_1[Y_1]; [X_2]S_2[Y_2]]\{Y\})
\triangleq (Y_2 := Y \cup Y_2); \text{ slice}([m_2]R_2[n_2], [X_2]S_2[Y_2]); \\
(Y_1 := Y_1 \cup X_1); \text{ slice}([m_1]R_1[n_1], [X_1]S_1[Y_1]); \\
X := X \cup X_1

\text{slice}\left(\left[\begin{array}{l}
[m] \text{if } B \text{ then } [m_1]R_1[n_1] \text{ else } [m_2]R_2[n_2]\{n\}, \\\n[X] \text{if } B \text{ then } [X_1]S_1[Y_1] \text{ else } [X_2]S_2[Y_2]\{Y\}\right]\right)
\triangleq (Y_1 := Y_1 \cup \text{clsr}(n_1, n, Y)); \text{ slice}([m_1]R_1[n_1], [X_1]S_1[Y_1]); \\
(Y_2 := Y_2 \cup \text{clsr}(n_2, n, Y)); \text{ slice}([m_2]R_2[n_2], [X_2]S_2[Y_2]); \\
X := X \cup \text{bslice}(B, m, m_1, X_1) \cup \text{bslice}(\neg B, m, m_2, X_2)

\text{slice}([m]\{\text{inv } i \text{ while } B \text{ do } [m_1]R[n_1]\}\{n\}, [X]\{\text{inv } K \text{ while } B \text{ do } [X_1]S[Y_1]\}\{Y\})
\triangleq \text{ do } \{ \begin{array}{l}
\text{ K } := K \cup \text{bslice}(\neg B, i, n, Y) \cup \text{bslice}(B, i, m_1, X_1); \\
(Y_1 := Y_1 \cup \text{clsr}(n_1, \iota, K)); \text{ slice}([m_1]R[n_1], [X_1]S[Y_1])
\end{array} \} \text{ until nothing changes;} \\
X := X \cup \text{clsr}(m, \iota, K)

\text{Figure 3: Abstract-value Slicing}
to backtrack the closure operator. Let \((n_0, I_0)\) be the result of the first slicing step. The slicer calls the subroutine \(\text{clsr}(m, n_0, I_0)\) to find out how the closure of \(m\) is used to show that the \(I_0\) entries of \(n_0\) are implied by \(m\). For instance, suppose that slice is given annotated loop body in Figure 2(a) and \([X] x_1 := x_2 + 1[Y]\) with \(X = \emptyset\) and \(Y = \{(2, 0), (1, 2)\}\). Then, it first eliminates \((1, 2)\) from \((2, 0), (1, 2)\), and modifies the \(x_1x_2\) entry of the “post-DBM” by \(\infty\). Then slice calls \(\text{clsr}\) with the pre-DBM, the modified post-DBM and the index-set \((2, 0)\). The \(x_2x_0\) entry of the modified post-DBM has value \(0\), while the corresponding entry of the pre-DBM has value \(\infty\). Thus, \(\text{clsr}\) concludes that the closure has been used in this implication, so it finds a shortest path \(x_2x_1x_0\) from \(x_2\) to \(x_0\) in the pre-DBM and returns the set \((2, 1), (1, 0)\) of entries in this path.

Suppose that \(\text{mPath}(m, (i, j))\) correctly calculates a shortest path from \(i\) to \(j\) in \(m\); when \(m\) does not contain a negative cycle, \(\text{mPath}(m, (i, j))\) returns a sequence \(i_1 \ldots i_n\) such that \(i_1 = i, i_n = j, 0 \leq i_1 \ldots i_n \leq N, \) and

\[
(\Sigma_{k=1}^{n-1} m_{i_ki_{k+1}}) = (m^*)_{ij}.
\]

Under this supposition, all of \(\text{clsr}, \text{bslice}, \) and \(\text{slice}\) are correct.

**Lemma 1** For all DBMs \(m, n\) and index sets \(I\), if \(m^* \subseteq n\), then

\[
(\text{prj}(m, \text{clsr}(m, n, I)))^* \subseteq \text{prj}(n, I).
\]

**Proof:** Let \(J\) be \(\text{clsr}(m, n, I)\), and let \(p\) and \(q\) be, respectively, \(\text{prj}(m, J)\) and \(\text{prj}(n, I)\). We only need to show that for all indices \((i, j)\), if \(q_{ij} \neq \infty\), then \((p^*)_{ij} \leq q_{ij}\). Consider an index \((i, j)\) such that \(q_{ij} \neq \infty\). Then, \((i, j)\) is in \(I\). So, \(q_{ij} = n_{ij}\). We now do the case analysis based on whether \(m_{ij} \leq n_{ij}\) or not. If \(m_{ij} \leq n_{ij}\), the index \((i, j)\) is in \(J\) by the definition of \(\text{clsr}\), and \(m_{ij} = p_{ij}\). Hence, \(p_{ij} \leq q_{ij}\). Moreover, by the definition of \(-^*\), we have \((p^*)_{ij} \leq p_{ij}\). Therefore, \((p^*)_{ij} \leq q_{ij}\), as required. On the other hand, if \(m_{ij} > n_{ij}\), by the definition of \(\text{clsr}\), \(J\) contains all the indices in a shortest path from \(i\) to \(j\). Thus, \((m^*)_{ij} = (p^*)_{ij}\). Since \(q_{ij} = n_{ij}\) and \((m^*)_{ij} \leq n_{ij}\), we have the required \((p^*)_{ij} \leq q_{ij}\).

**Lemma 2** For all DBMs \(m, n\), boolean expressions \(B\), and index sets \(I\), if \([B]m \subseteq n\), then

\[
\left([B] \text{prj}(m, \text{bslice}(B, m, n, I))\right) \subseteq \text{prj}(n, I).
\]

**Proof:** Let \(J\) be \(\text{bslice}(B, m, n, I)\). We will use induction on the structure of \(B\) to prove

\[
\left([B] \text{prj}(m, \text{bslice}(B, m, n, I))\right) \subseteq \text{prj}(n, I).
\]

1. **Case \(B \equiv x_i - x_j \leq c\)**: In this case, we only need to show that

\[
\forall (k, l) \in I. \ (\lfloor x_i - x_j \leq c \rfloor \text{prj}(m, J))_{kl} \leq (\text{prj}(n, I))_{kl}.
\]

Let \((k, l)\) be an index in \(I\). If \((k, l) = (j, i)\) and \(n_{kl} \geq c\), then \((k, l) \notin J\), and so,

\[
(\lfloor x_i - x_j \leq c \rfloor \text{prj}(m, J))_{kl} = \min(\infty, c) = c \leq n_{kl} = (\text{prj}(n, I))_{kl}.
\]

On the other hand, if \((k, l) \neq (j, i)\) or \(m'_{kl} \leq c\), then \((k, l) \in J\), and so, \((\lfloor x_i - x_j \leq c \rfloor \text{prj}(m, J))_{kl}\) and \((\lfloor x_i - x_j \leq c \rfloor m)_{kl}\) are equal. Thus,

\[
(\lfloor x_i - x_j \leq c \rfloor \text{prj}(m, J))_{kl} = (\lfloor x_i - x_j \leq c \rfloor m)_{kl} \leq n_{kl} = (\text{prj}(n, I))_{kl}.
\]
2. Case $B \equiv B_1 \vee B_2$: We prove the lemma for this case as follows:

$[B_1 \vee B_2]([\text{prj}(m, J)]) \subseteq [B_1]([\text{prj}(m, J)]) \cup [B_2]([\text{prj}(m, J)])$,

$\subseteq [B_1]([\text{prj}(m, \text{bslice}(B_1, m, n, I))] \cup [B_2]([\text{prj}(m, \text{bslice}(B_2, m, n, I))]$

$\subseteq \text{prj}(n, I) \cup \text{prj}(n, I) \quad (\because \text{induction hypo.})$

$= \text{prj}(n, I)$

3. Case $B \equiv B_1 \wedge B_2$: Define DBMs $m_1, m_2$, and the sets $I_1, I_2, J_1, J_2$ of indices as follows:

$m_1 = [B_1][m \ I_1] = \{(i, j) \in I \ | \ (m_1)_{ij} \leq n_{ij}\} \quad J_1 = \text{bslice}(B_1, m, n \cup m_1, I_1)$

$m_2 = [B_2][m \ I_2] = \{(i, j) \in I \ | \ (m_2)_{ij} \leq n_{ij}\} \quad J_2 = \text{bslice}(B_2, m, n \cup m_2, I_2)$

Then, $J = J_1 \cup J_2$. Hence, $[\text{prj}(m, J)] \subseteq [\text{prj}(m, J_1)]$ and $[\text{prj}(m, J_2)] \subseteq [\text{prj}(m, J_2)]$. We also note that $I = I_1 \cup I_2$, because $m_1 \cap m_2 \subseteq n$. Using these facts, we prove the required order relationship as follows:

$[B_1 \wedge B_2]([\text{prj}(m, J)]) \subseteq [B_1]([\text{prj}(m, J)]) \cap [B_2]([\text{prj}(m, J)]) \quad (\because \text{def. of } \cap)$

$\subseteq [B_1]([\text{prj}(m, J_1)]) \cap [B_2]([\text{prj}(m, J_2)]) \quad (\because \text{def. of } [B_1])$

$= \text{prj}(n, I_1) \cap \text{prj}(n, I_2) \quad (\because \text{def. of } I)$

$\subseteq \text{prj}(n, I)$

Therefore, $\text{prj}(n, I) \subseteq \text{prj}(n, I)$.

$\square$

**Theorem 3 (Correctness)** Suppose that the abstract-value slicer is given $(A, T)$ such that $A$ satisfies the constraints generated by the algorithm $C$, and $T$ and $A$ have the same underlying program. Then, $\text{slice}(A, T)$ terminates, and the final value $T_1$ of $T$ satisfies the following properties:

1. For each variable $X$ in $T$, the final value of $X$ includes the initial value of $X$.
2. When we project every annotation in $A$ by the corresponding index set in $T_1$, the resulting annotated program satisfies the constraints generated by $C$.

**Proof:** All the updates of variables in $\text{slice}(A, T)$ have the form of $X := X \cup -$. Thus, as the execution of $\text{slice}(A, T)$ proceeds, the value of each variable in $T$ gets increased. Note that this increasing value of a variable eventually becomes stabilized, because the variable cannot have a greater than $(N + 1) \times (N + 1)$. This change of variables in $T$ shows that all the do-until loops in $\text{slice}(A, T)$ terminate, and so does $\text{slice}(A, T)$ itself, and the final value of each variable in $T$ includes the initial value of the variable. Let $T_1$ be the index-set annotated program that records the final value of each variable in $T$. We will now prove that when every annotation in $A$ is projected by the corresponding index set in $T_1$, the resulting annotated program satisfies the constraints generated by $C$. For this proof, we use induction on the structure of $A$.

**Case** $A \equiv [m][\text{skip}[n]]$: In this case, there exist index sets $I$ and $J$ such that the final value of $T$ is $[J \cup I][\text{skip}[I]]$. Since $m \subseteq n$, the required constraint holds by the following reason:

$\text{prj}(m, J \cup I) \subseteq \text{prj}(m, I) \quad (\because \text{prj}(m, -) \text{ is anti-monotone})$

$\subseteq \text{prj}(n, I) \quad (\because \text{prj}(-, I) \text{ is monotone})$
Now, consider the other case:
as follows:

By the definition of the abstract-value slicer, there exist index sets \( A \) such that the final value of \( T \) is \([J \cup I]_{x_i := x_i + cI} \). Since \( A \) satisfies the constraints in \( \mathcal{C}(A) \),

\[
\left( m[(k, i) \rightarrow m_{ki} + c, (i, k) \rightarrow m_{ik} - c]_{0 \leq k \neq i \leq N} \right) \subseteq n.
\]

Using this order relationship, we will show the required

\[
\left( \text{prj}(m, J \cup I)[(k, i) \rightarrow m_{ki} + c, (i, k) \rightarrow m_{ik} - c]_{0 \leq k \neq i \leq N} \right) \subseteq \text{prj}(n, I).
\]

Let \( p \) be the DBM on the left-hand side of the above order relationship, and let \((k, l)\) be a DBM index. If \((k, l) \notin I\), then \( \text{prj}(n, I)_{kl} = \infty \), and so, \( p_{kl} \leq \text{prj}(n, I)_{kl} \). Otherwise,

\[
p_{kl} = \left( m[(k, i) \rightarrow m_{ki} + c, (i, k) \rightarrow m_{ik} - c]_{0 \leq k \neq i \leq N} \right)_{kl} \quad (\because (k, l) \in I)
\]

\[
\leq n_{kl} \quad (\because A \text{ satisfies the constraints in } \mathcal{C}(A))
\]

\[
\leq \text{prj}(n, I)_{kl} \quad (\because (k, l) \in I)
\]

**Case** \( A \equiv [m]_{x_i := x_j + c[n]} \) **where** \( i \neq j \): By the definition of slice, there exist index sets \( I, J \) such that the final value of \( T \) is

\[
\left[ J \cup \left[ \text{clsr}(m, n)((i, k) \rightarrow \infty, (k, i) \rightarrow \infty)_{0 \leq k \leq N}, I - \{(i, k), (k, i) \mid 0 \leq k \leq N\} \right] \right]_{x_i := x_j + c[I]}.
\]

Let \( I' \) be \( I - \{(i, k), (k, i) \mid 0 \leq k \leq N\} \), let \( n' \) be \( n[(i, k) \rightarrow \infty, (k, i) \rightarrow \infty]_{0 \leq k \leq N} \), and let \( J' \) be \( \text{clsr}(m, n', I') \). We will prove that

\[
\left( \text{prj}(m, J \cup J')^*([(k, i) \rightarrow \infty, (i, k) \rightarrow \infty]_{0 \leq k \neq i \leq N})[(j, i) \rightarrow c, (i, j) \rightarrow -c] \right) \subseteq \text{prj}(n, I).
\]

Let \( p \) be the DBM on the left-hand side of the above order relationship. Pick a DBM index \((k, l)\). If \((k, l)\) is not in \( I \), then \( \text{prj}(n, I)_{kl} = \infty \). So, \( p_{kl} \leq \text{prj}(n, I)_{kl} \). Otherwise, \( \text{prj}(n, I)_{kl} = n_{kl} \). So, it suffices to show that \( p_{kl} \leq n_{kl} \). We do the case analysis based on whether \( k = i \lor l = i \) holds or not. Let’s first consider the case that \( k = i \lor l = i \). By assumption, \( A \) satisfies the following unique constraint in \( \mathcal{C}(A) \):

\[
\left( m^*[([(k, i) \rightarrow \infty, (i, k) \rightarrow \infty]_{0 \leq k \neq i \leq N})[(j, i) \rightarrow c, (i, j) \rightarrow -c] \right) \subseteq n.
\]

Let \( q \) be the DBM on the left-hand side of the above constraint. Then, the constraint implies that \( q_{kl} \leq n_{kl} \). From this inequality, we can derive the required \( p_{kl} \leq n_{kl} \), because

\[
\forall(k', l'), (k' = i \lor l' = i) \Rightarrow (q_{k' l'} = p_{k' l'}).
\]

Now, consider the other case: \( k \neq i \) and \( l \neq i \). In this case, we can prove the required inequality as follows:

\[
p_{kl} = \begin{cases} \text{prj}(m, J \cup \text{clsr}(m, n', I'))_{kl} & (\because k \neq i \text{ and } l \neq i) \\ \text{prj}(m, \text{clsr}(m, n', I'))_{kl} & (\because \text{prj}(m, -) \text{ is anti-monotone}) \\ \text{prj}(n', I')_{kl} & (\because \text{Lemma 1 and } m^* \subseteq n \subseteq n') \\ n_{kl} & (\because (k, l) \in I') \end{cases}
\]

**Case** \( A \equiv [m][[m_1]_{R_1[n_1]}; [m_2]_{R_2[n_2]}][n] \): In this case, \( T = [X][T_1; T_2][Y] \) for some \( T_1, T_2, X, Y \). By the definition of the abstract-value slicer, there exist index sets \( I, I_1, I_2, J, J_1, J_2 \), and programs \( S_1, S_2 \) annotated with index sets such that the final value of \( T \) is

\[
[J \cup J_1][[J_1]_{S_1[I_1 \cup J_2]}; [J_2]_{S_2[I_2 \cup I]}][I].
\]
Both \([J_1]S_1[I_1 \cup J_2]\) and \([J_2]S_2[I_2 \cup I]\) are the final values of \(T_1\) and \(T_2\), respectively, right after the recursive calls, \(\text{slice}([m_1]R_1[m_1], T_1)\) and \(\text{slice}([m_2]R_2[m_2], T_2)\). Thus, by induction hypothesis, all the constraints for \(R_1\) and \(R_2\) hold. We now show that the following remaining constraints also hold:

\[
\text{prj}(m, J \cup J_1) \subseteq \text{prj}(m_1, J_1), \quad \text{prj}(n_1, I_1 \cup J_2) \subseteq \text{prj}(m_2, J_2), \quad \text{prj}(n_2, I_2 \cup I) \subseteq \text{prj}(n, I).
\]

Since all the constraints in \(C(A)\) hold, we have

\[
m \subseteq m_1 \land n_1 \subseteq m_2 \land n_2 \subseteq n.
\]

These order relationships implies that the required constraints hold, because \(\text{prj}\) is anti-monotone on the second argument and monotone on the first argument.

**Case** \(A \equiv [m][\text{if } B \text{ then } [m_1]R_1[n_1] \text{ else } [m_2]R_2[n_2]][n]\): In this case, there exist variables \(X, Y\) and programs \(T_1, T_2\) annotated with variables such that \(T = [X][\text{if } B \text{ then } T_1 \text{ else } T_2][Y]\).

By the definition of the abstract-value slicer, there exist index sets \(I, I_1, I_2, J, J_1, J_2\), and programs \(S_1, S_2\) annotated with index sets such that the final value of \(T\) is

\[
[J \cup \text{bslice}(B, m, m_1, J_1) \cup \text{bslice}(\neg B, m, m_2, J_2)]
\]

\[
\text{if } B \text{ then } [J_1]S_1[I_1 \cup \text{clsr}(n_1, n, I)] \text{ else } [J_2]S_2[I_2 \cup \text{clsr}(n_2, n, I)]
\]

By the definition of slice, for \(i = 1, 2\), annotated program \([J_i]S_i[I_i \cup \text{clsr}(n_i, n, I)]\) is the value of \(T_i\) right after the recursive call \(\text{slice}([m_1]R_i[n_i], T_i)\). Thus, by the induction hypothesis, among the constraints in \(C(A)\), those about the true and false branches hold. We now show that the following remaining four constraints are also true:

\[
[B] \text{prj}(m, J \cup \text{bslice}(B, m, m_1, J_1) \cup \text{bslice}(\neg B, m, m_2, J_2)) \subseteq \text{prj}(m_1, J_1)
\]

\[
[\neg B] \text{prj}(m, J \cup \text{bslice}(B, m, m_1, J_1) \cup \text{bslice}(\neg B, m, m_2, J_2)) \subseteq \text{prj}(m_2, J_2)
\]

\[
\text{prj}(n_1, I_1 \cup \text{clsr}(n_1, n, I)) \subseteq \text{prj}(n_1, I)
\]

\[
\text{prj}(n_2, I_2 \cup \text{clsr}(n_2, n, I)) \subseteq \text{prj}(n_1, I)
\]

We use the fact that all the constraints in \(C(A)\) hold. In particular, we use the following order relationships:

\[
[B] m \subseteq m_1, \quad [\neg B] m \subseteq m_2, \quad (n_1)^* \subseteq n, \quad (n_2)^* \subseteq n.
\]

From these relationships, the validity of the required four constraints follows; since \(\text{prj}\) is anti-monotone on the second argument, the constraints are implied by the following four order relationships:

\[
[B] \text{prj}(m, \text{bslice}(B, m, m_1, J_1)) \subseteq \text{prj}(m_1, J_1)
\]

\[
[\neg B] \text{prj}(m, \text{bslice}(\neg B, m, m_2, J_2)) \subseteq \text{prj}(m_2, J_2)
\]

\[
\text{prj}(n_1, \text{clsr}(n_1, n, I))^* \subseteq \text{prj}(n_1, I)
\]

\[
\text{prj}(n_2, \text{clsr}(n_2, n, I))^* \subseteq \text{prj}(n_1, I)
\]

and these order relationships hold because of Lemma 2 and 1.

**Case** \(A \equiv [m][\text{inv } l][\text{while } B \text{ do } [m_1]R_1[n_1]][n]\): In this case, there are \(X, Y, K, T_1\) such that \(T = [X][\text{while } B \text{ do } T_1][Y]\). By the definition of slice, there exist index sets \(I, I_1, J, J_1, L\) and program \(S_1\) annotated with index sets such that

1. the final value of \(T\) is \([J \cup \text{clsr}(m, t, L)][\text{inv } l][\text{while } B \text{ do } [J_1]S_1[I_1]][I]\);
2. \(L = L \cup \text{bslice}(\neg B, t, n, I) \cup \text{bslice}(B, t, m_1, J_1);\) and
3. \(I_1 = I_1 \cup \text{clsr}(n_1, t, L)\).
<table>
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<th>useful</th>
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<th>slicing time</th>
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<td>203</td>
<td>1047</td>
<td>84%</td>
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</table>

*a* number of variables

*b* number of program labels; the labels are given to each command

*c* DBM entries with value $m_{ij}$ such that $i \neq j$ and $m_{ij} \neq \infty, -\infty$

*d* Partition function in Quicksort

*e* Knuth-Morris-Pratt pattern matching algorithm

Table 1: Number of Sliced DBM Entries

Then, the value of $T_1$ right after the last recursive call is $[J_1]S_1[I_1]$. Thus, $[J_1]S_1[I_1]$ satisfies all the constraints generated by $C$. We now need to show that the following four constraints are true:

- $prj(m, J \cup clsr(m, \iota, L))^* \sqsubseteq prj(i, L)$
- $prj(n_1, I_1)^* \sqsubseteq prj(i, L)$
- $[B]prj(i, L) \sqsubseteq prj(m_1, J_1)$
- $[\neg B]prj(i, L) \sqsubseteq prj(n, I)$

We “simplify” these four constraints using two facts: $prj$ is anti-monotone on the second argument, and $L$ and $I_1$ are solutions for the equations:

$$L = L \cup bslice(\neg B, \iota, n, I) \cup bslice(B, \iota, m_1, J_1) \quad \text{and} \quad I_1 = I_1 \cup clsr(n_1, \iota, L).$$

Because of the two facts, the above four constraints are implied by the following:

- $prj(m, clsr(m, \iota, L))^* \sqsubseteq prj(i, L)$
- $prj(n_1, clsr(n_1, \iota, L))^* \sqsubseteq prj(i, L)$
- $[B]prj(i, bslice(B, \iota, m_1, J_1)) \sqsubseteq prj(m_1, J_1)$
- $[\neg B]prj(i, bslice(\neg B, \iota, n, I)) \sqsubseteq prj(n, I)$

Note that these are precisely the correctness statements of $clsr$ and $bslice$ in Lemma 2 and 1. These correctness statements hold, because the conditions for the correctness of $clsr$ and $bslice$, namely,

$$m^* \sqsubseteq \iota, \quad (n_1)^* \sqsubseteq \iota, \quad [B]_l \sqsubseteq m_1, \quad \text{and} \quad [\neg B]_l \sqsubseteq n$$

are satisfied by the assumption on the constraints in $C(A)$. □

4 Experimental Results

We tested the efficiency of the abstract-value slicer in the context of proof construction. We implemented Miné’s abstract interpreter, the abstract-value slicer, and the proof construction algorithm in our previous work [SYY03]. In our experiment, we first ran Miné’s abstract interpreter with five array accessing programs, and obtained approximate invariants which are strong enough to show the absence of array bounds errors. Then, we ran the slicer for each of the computed abstract interpretation results, and measured how many invariants in the result have been eliminated. Finally, we applied the proof construction algorithm to both the original abstract interpretation results and their sliced versions, and measured how much the slicer reduces the size of the constructed proofs.
Table 2: Reduction in the Proof Size

Table 1 shows how many invariants have been sliced out by the abstract-value slicer. The fourth column, labeled by “live”, contains the number of all the nontrivial DBM entries in the result of the abstract interpreter, and the sixth column labeled by “useless” shows how many of those nontrivial entries the slicer found useless for verifying the absence of array bounds errors. The experimental result shows that about 63% to 84% of computed invariants are not needed for the verification.

The reduction in the size of constructed proofs is shown in Table 2. For each constructed proof, we counted the number of used Hoare logic rules, and that of used first-order logic rules. The abstract-value slicer did not reduce the number of used Hoare logic rules, because Hoare rules are applied as many times as the number of command constructs in a program, and the abstract-value slicer does not change the program. However, the slicer reduced the number of used first-order logic rules. The experimental result shows that about 56% to 88% less rules are used for showing implications between first-order logic formulas.

5 General Framework

We now generalize the results in Section 3, and propose a framework for constructing a correct abstract-value slicer.

Consider an abstract interpreter $T$ with the following data:

- An abstraction domain $(\mathcal{A}, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$, which is a distributive lattice.$^3$
- Concretization and abstraction maps, $\gamma$ and $\alpha$.
- An abstract semantics $\llbracket - \rrbracket$ for all atomic “terms”: for all assignments $x := E$ and inequalities $E \leq E'$, both $\llbracket x := E \rrbracket$ and $\llbracket E \leq E' \rrbracket$ are defined, and denote monotone functions on $\mathcal{A}$.

In order to construct an abstract-value slicer for $T$ in our framework, we need to provide two kinds of data. First, for each $a \in \mathcal{A}$, we need to specify a finite sub-poset $\mathcal{A}_a$ of $\mathcal{A}$ and a monotone mapping $\alpha_a : \mathcal{A} \rightarrow \mathcal{A}_a$ such that

1. $\mathcal{A}_a$ is a lattice such that the least element of $\mathcal{A}_a$ is $a$, and all the other lattice operations of $\mathcal{A}_a$ are inherited from $\mathcal{A}$;
2. when inc is the inclusion from $\mathcal{A}_a$ to $\mathcal{A}$, the quadruple $(\mathcal{A}, \mathcal{A}_a, \alpha_a, \text{inc})$ is a Galois injection;

$^3$A lattice $(\mathcal{A}, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ is distributive if and only if

$$\forall a, b, d \in A. a \sqcup (b \sqcap d) = (a \sqcup b) \sqcup (a \sqcap d) \land a \sqcap (b \sqcup d) = (a \sqcap b) \sqcup (a \sqcup d).$$
3. \( \alpha_a \) preserves the binary meet \( \sqcap \); and
4. for all \( a, b \in \mathcal{A} \), if \( a \subseteq b \), then for all \( a' \in \mathcal{A}_a \),

\[
a' \sqcup b \in \mathcal{A}_b \quad \text{and} \quad a' \sqcap b = \alpha_b(a').
\]

Intuitively, \( \mathcal{A}_a \) denotes the set of all “sub-information” of \( a \), and \( \alpha_a(b) \) extracts the strongest sub-information of \( a \) that is implied by \( b \). For example, in the abstract-value slicer in Section 3, the lattice \( \mathcal{A}_m \) consists of all DBMs \( n \) that can be obtained by dropping some entries in \( m \): \( \forall ij. m_{ij} \neq n_{ij} \Rightarrow n_{ij} = \infty \). And \( \alpha_m(n) \) records which entries of \( m \) are implied by the corresponding entries of \( n \):

\[
(\alpha_m(n))_{ij} = \begin{cases} 
\infty & \text{if } m_{ij} < n_{ij} \\
m_{ij} & \text{otherwise}
\end{cases}
\]

Second, we need to define a procedure \( \text{btrack} \) that calculates the “backtracking” of forward abstract computation. The procedure \( \text{btrack} \) takes three parameters. The first parameter is a monotone function \( f \) on \( \mathcal{A} \), and it always has one of following three forms: \( [x := E], [E \leq E'] \), and the lower closure \( (\alpha \circ \gamma) \). The other two parameters are abstract elements \( a \) and \( b \) such that \( b \) approximates the result of \( f \) from \( a \): \( f(a) \subseteq b \). Given such \( f, a, b \), procedure \( \text{btrack} \) returns a monotone function of type \( \mathcal{A}_b \to \mathcal{A}_a \) such that\(^4\)

\[
\forall b' \in \mathcal{A}_b, f(\text{btrack}(f, a, b)(b')) \subseteq b'.
\]

Here the input \( b' \) to \( \text{btrack}(f, a, b) \) indicates which parts of \( b \) are used for verification, and \( \text{btrack}(f, a, b) \) denotes which parts of \( a \) are necessary to compute \( b' \).

Figure 4 shows an abstract-value slicer \( \text{gslice} \) for \( T \) in an imperative style, and its two subroutines \( \text{order} \) and \( \text{gbslice} \). Note that algorithm \( \text{gbslice} \) and \( \text{gslice} \) generalize \( \text{bslice} \) and \( \text{slice} \) in Section 3, respectively; in \( \text{bslice} \) and \( \text{slice} \), we use index sets to represent elements in \( \mathcal{A}_a \). We state the correctness of \( \text{order} \), \( \text{gbslice} \) and \( \text{gslice} \):

**Lemma 4** For all abstract values \( a \) and \( b \), if \( a \subseteq b \), then \( \lambda b'. \text{order}(a, b, b') \) is a well-defined monotone function from \( \mathcal{A}_b \to \mathcal{A}_a \) such that

\[
\forall b' \in \mathcal{A}_b, \text{order}(a, b, b') \subseteq b'.
\]

**Proof:** We first show that \( \text{order}(a, b, b') \) is a well-defined element in \( \mathcal{A}_a \). Since \( \mathcal{A}_a \) is finite and \( \mathcal{A} \) is a lattice, there exists the join of \( \{ a' \in \mathcal{A}_a \mid \alpha_b(a') \subseteq b' \} \) in \( \mathcal{A} \). Thus, \( \text{order}(a, b, b') \) is a well-defined element in \( \mathcal{A} \). To show that \( \text{order}(a, b, b') \) is indeed in \( \mathcal{A}_a \), we only need to show that \( \{ a' \in \mathcal{A}_a \mid \alpha_b(a') \subseteq b' \} \) is nonempty, because the binary join of \( \mathcal{A}_a \) is inherited from \( \mathcal{A} \). By assumption, \( a \subseteq b \), and since \( b \) is the least element in \( \mathcal{A}_b, b \subseteq b' \). Thus, \( a \subseteq b' \). Now, since \( b' \in \mathcal{A}_b \), by the Galois injection, \( \alpha_b(a') \subseteq b' \).

The condition about \( \text{order}(a, b, b') \), that is, \( \text{order}(a, b, b') \subseteq b' \), follows from the definition of \( \text{order} \). Since \( b' \) is the upper bound of \( \{ a' \in \mathcal{A}_a \mid \alpha_b(a') \subseteq b' \} \), it should be greater than or equal to the least upper bound \( \text{order}(a, b, b') \).

The monotonicity of \( \text{order}(a, b, -) \) also follows from the definition of \( \text{order} \). If \( b' \subseteq b'' \), then we have \( \{ a' \in \mathcal{A}_a \mid \alpha_b(a') \subseteq b' \} \subseteq \{ a' \in \mathcal{A}_a \mid \alpha_b(a') \subseteq b'' \} \). So, the least upper bound \( \text{order}(a, b, b') \) for the first set is less than or equal to the least upper bound \( \text{order}(a, b, b'') \) for the other set. \( \square \)

**Lemma 5** Let \( a, b \) be abstract elements and let \( B \) be a boolean expression such that \( [B]a \subseteq b \). Then, \( \lambda b'. \text{gbslice}(B, a, b, b') \) is a well-defined monotone function from \( \mathcal{A}_b \to \mathcal{A}_a \) such that

\[
\forall b' \in \mathcal{A}_b, [B](\text{gbslice}(B, a, b, b')) \subseteq b'.
\]

\(^4\)Here we omitted the applications of the inclusion maps.
Proof: We use the induction on the structure of B to prove this lemma.

1. Case $B \equiv E \leq E'$: This case follows from the assumption on $btrack$.

2. Case $B \equiv B_1 \lor B_2$: By the induction hypothesis, $\lambda b'. \mathrm{gbslice}(B_1, a, b, b')$ and $\lambda b'. \mathrm{gbslice}(B_2, a, b, b')$ are monotone functions from $A_b$ to $A_a$. Moreover, since the binary meets of $A_a$ and $A$ are identical, $\lambda(a'_1, a'_2). a'_1 \sqcap a'_2$ is a monotone function from $A_a \times A_a$ to $A_a$. Since $\lambda b'. \mathrm{gbslice}(B_1 \lor B_2, a, b, b')$ is constructed by composing these well-defined monotone functions, it is also a well-defined monotone function from $A_b$ to $A_a$. We now show that

$$\forall b' \in A_b, [B_1 \lor B_2](\mathrm{gbslice}(B_1 \lor B_2, a, b, b')) \sqsubseteq b'.$$

For all $b'$ in $A_b$, we show the required order relationship as follows:

$$[B_1 \lor B_2](\mathrm{gbslice}(B_1 \lor B_2, a, b, b'))$$

$\equiv (\because \text{the definition of } [B_1 \lor B_2])$

$$[B_1](\mathrm{gbslice}(B_1 \lor B_2, a, b, b')) \sqcup [B_2](\mathrm{gbslice}(B_1 \lor B_2, a, b, b'))$$

$\sqsubseteq (\because \text{gbslice}(B_i, a, b, b') \sqsubseteq \mathrm{gbslice}(B_i, a, b, b') \text{ for } i = 1, 2)$

$$[B_1](\mathrm{gbslice}(B_1, a, b, b')) \sqcup [B_2](\mathrm{gbslice}(B_2, a, b, b'))$$

$\sqsubseteq (\because \text{the induction hypothesis})$

$$b' \sqcup b'$$

$\equiv b'$

3. Case $B \equiv B_1 \land B_2$: Let $b_1$ be $[B_1]b$ and let $b_2$ be $[B_2]b$. Then, all of the following four functions are well-defined and monotone, and have the specified type:

$$\lambda b'. \alpha_{b_1\land b_2}(b' \sqcup \alpha_1(b_1)) : A_b \rightarrow A_{b_1\land b_2}$$

$$\lambda b'. \alpha_{b_1\land b_2}(b' \sqcup \alpha_2(b_2)) : A_b \rightarrow A_{b_1\land b_2}$$

$$\lambda b'. \mathrm{gbslice}(B_1, a, b \sqcup b_1, b') : A_{b_1\land b_2} \rightarrow A_a$$

$$\lambda b'. \mathrm{gbslice}(B_2, a, b \sqcup b_2, b') : A_{b_1\land b_2} \rightarrow A_a$$

$$\lambda(a'_1, a'_2), a'_1 \sqcap a'_2 : A_a \times A_a \rightarrow A_a$$

Since $\lambda b'. \mathrm{gbslice}(B_1 \land B_2, a, b, b')$ is defined by composing the above functions, it is a well-defined monotone function from $A_b$ to $A_a$. Now, it remains to show that for all $b' \in A_b$,

$$[B_1 \land B_2](\mathrm{gbslice}(B_1 \land B_2, a, b, b')) \sqsubseteq b'.$$

Pick $b'$ from $A_b$. Let $b'_1 = \alpha_{b_1\land b_2}(b' \sqcup \alpha_1(b_1))$ and $b'_2 = \alpha_{b_1\land b_2}(b' \sqcup \alpha_2(b_2))$. We show the required
order relationship as follows:

\[
[B_1 \land B_2](\text{gbslice}(B_1 \land B_2, a, b, b')) = \\
\begin{cases}
\vdash \text{the definition of } [B_1 \land B_2] \\
[B_1](\text{gbslice}(B_1 \land B_2, a, b, b')) \sqcap [B_2](\text{gbslice}(B_1 \land B_2, a, b, b')) \\
\end{cases}
\]

\[
\begin{array}{c}
\begin{cases}
\vdash \text{gbslice}(B_1 \land B_2, a, b, b') \sqsubseteq \text{gbslice}(B_i, a, b \sqcup b_i, b_i') \text{ for all } i = 1, 2 \\
\end{cases}
\end{array}
\]

\[
\begin{array}{c}
\vdash \text{the induction hypothesis} \\
b_1 \sqcap b_2 \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \text{the definition of } b_1 \text{ and } b_2 \\
\alpha_{b_1, b_2}(b' \sqcup \alpha_b(b_1)) \sqcap \alpha_{b_2, b_2}(b' \sqcup \alpha_b(b_2)) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash b \sqsubseteq b \sqcup b_i, (b' \sqcup \alpha_b(b_1)) \in A_b, \text{ and the fourth condition on } A_a \text{ and } A_a \\
(b' \sqcup \alpha_b(b_1)) \sqcap b \sqcup b_1 \sqcap (b' \sqcup \alpha_b(b_2)) \sqcap b \sqcup b_2 \\
\end{array}
\]

\[
\begin{array}{c}
\vdash b' \in A_b \text{ and } b \text{ is the least element in } A_b; \text{ so, } b \sqsubseteq b' \\
(b' \sqcup \alpha_b(b_1)) \sqcap (b' \sqcup \alpha_b(b_2)) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash (A, A_b, \alpha_b, \text{inc}) \text{ is a Galois injection; so, } b_i \sqsubseteq \alpha_b(b_i) \text{ for } i = 1, 2 \\
(b' \sqcup \alpha_b(b_1)) \sqcap (b' \sqcup \alpha_b(b_2)) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash A \text{ is distributive} \\
(b' \sqcap b) \sqcup (\alpha_b(b_1) \sqcap b') \sqcup (b' \sqcap \alpha_b(b_2)) \sqcup (\alpha_b(b_1) \sqcap \alpha_b(b_2)) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \alpha_b \text{ preserves the binary meet } \sqcap \\
(b' \sqcap b) \sqcup (\alpha_b(b_1) \sqcap b') \sqcup (b' \sqcap \alpha_b(b_2)) \sqcup \alpha_b(b_1) \sqcap \alpha_b(b_2) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash b = b_1 \sqcap b_2 \text{ and } b' \in A_b; \text{ so, } \alpha_b(b_1) \sqcap b_2 = \alpha_b(b) \sqsubseteq b' \\
\end{array}
\]

Definition 6 (Slicing Domain) Let A be a program annotated with abstract elements in A. The slicing domain A_A for A consists of annotated programs A' such that

1. A and A' have the same underlying program; and

2. for all program point, if a and a' are, respectively, the annotations in A and A' at that point, then a' \in A_a.

The slicing domain A_A is ordered pointwise: for A', A'' \in A_A, A' \sqsubseteq A'' iff for all program point, the annotation a' in A' at the point is less than or equal to the annotation a'' in A'' at the same program point.

Theorem 7 (Correctness) Suppose that the abstract-value slicer is given (A, T) such that A satisfies the constraints generated by an algorithm C(A) in Appendix A, and the initial value A' of T is in the slicing domain A_A. Then, gslice(A, T) terminates, and the final value A'' of T satisfies the following properties:

1. A'' is in the slicing domain A_A;

2. A'' \sqsubseteq A'; and

3. A'' satisfies the constraints in C(A'') in Appendix A.

Proof: All the updates of variables X in gslice have the form of X := X \sqcap a' for some a'. Thus, as the algorithm gslice proceeds, the values of all the variables in T get smaller. Moreover, all the updates in gslice preserve the invariant that the value of T, which is an annotated program, is in the slicing domain A_A. Thus, as the algorithm gslice proceeds, T's value decreases in the slicing domain A_A. Since for each a \in A, A_a is finite, the slicing domain A_A is finite. Thus, T cannot decrease forever, and should become stable. This property of T's value shows that all the iterations in gslice eventually terminate, so the whole algorithm gslice terminates; and that
when \( \text{gslice} \) terminates, \( T \)'s value is in the slicing domain \( \mathcal{A}_A \), and it is smaller than or equal to \( T \)'s initial value. We will now show that \( T \)'s final value satisfies the constraints generated by \( C \), by using induction on the structure of \( A \).

Case \( A \equiv [a]\text{skip} \): Then, there exist \( a' \in \mathcal{A}_a \) and \( b' \in \mathcal{A}_b \) such that \([a' \sqcap \text{order}(a, b, b')] \text{skip} [b']\) is the final value of \( T \). Then,
\[
a' \sqcap \text{order}(a, b, b') \subseteq \text{order}(a, b, b') \subseteq b' \quad (\because \text{Lemma 4}).
\]

Case \( A \equiv [a]x := E[b] \): Let \( T = [X](T_1; T_2)[Y] \). Then, there exist \( a' \in \mathcal{A}_a \) and \( b' \in \mathcal{A}_b \) such that the final value of \( T \) is
\[
[a' \sqcap \text{btrack}([x := E], a, b, b')][x := E[b']].
\]

Then,
\[
a' \sqcap \text{btrack}([x := E], a, b, b') \subseteq \text{btrack}([x := E], a, b, b') \subseteq b' \quad (\because \text{the assumption on btrack}).
\]

Case \( A \equiv [a][\{a_1\}R_1[b_1]; [a_2]R_2[b_2]) [b] \): Let \( T_1, T_2 \) be programs annotated with variables such that \( T = [X]T_1; T_2[Y] \). By the definition of \( \text{gslice} \), there exist
\[
a' \in \mathcal{A}_a, \quad a_1' \in \mathcal{A}_{a_1}, \quad a_2' \in \mathcal{A}_{a_2}, \quad b' \in \mathcal{A}_b, \quad b_1' \in \mathcal{A}_{b_1}, \quad b_2' \in \mathcal{A}_{b_2},
\]
and \( R_1', R_2' \) such that such that the final value of \( T \) is
\[
[a' \sqcap \text{order}(a, a_1, a_1') \sqcap [a_1' \sqcap \text{order}(b_1, a_2, a_2') \sqcap [a_2' \sqcap \text{order}(b_2, b, b')])][b'],
\]
and \([a_1' \sqcap \text{order}(b_1, a_2, a_2')] \) and \([a_2' \sqcap \text{order}(b_2, b, b')] \) are the final values of \( T_1 \) and \( T_2 \), respectively. Here, \( a', a_1', a_2', b', b_1', b_2', \) and \( b' \) are the initial values of the variables at the corresponding program points; and \( a_1' \) and \( a_2' \) are the final values of the corresponding variables. We can represent the final value of \( T \) in this way, because \( \text{gslice}(A, T) \) never modifies the last variable in \( T \). Since \([a_1' \sqcap \text{order}(b_1, a_2, a_2')] \) and \([a_2' \sqcap \text{order}(b_2, b, b')] \) are also the values of \( A_1 \) and \( A_2 \) right after the recursive calls, by the induction hypothesis, their annotations satisfy all the constraints generated by \( C \). So, we can focus on the following three constraints:
\[
a' \sqcap \text{order}(a, a_1, a_1') \subseteq a_1', \quad b_1' \sqcap \text{order}(b_1, a_2, a_2') \subseteq a_2', \quad b_2' \sqcap \text{order}(b_2, b, b') \subseteq b'.
\]
All these constraints hold by the “soundness” of \( \text{order} \) (Lemma 4).

Case \( A \equiv [a](\text{if } B \text{ then } [a_1]R_1[b_1] \text{ else } [a_2]R_2[b_2]) [b] \): Let \( T_1, T_2 \) be programs annotated with variables such that \( T = [X]\text{if } B \text{ then } T_1 \text{ else } T_2[Y] \). By the definition of \( \text{gslice} \), there exist
\[
a' \in \mathcal{A}_a, \quad a_1' \in \mathcal{A}_{a_1}, \quad a_2' \in \mathcal{A}_{a_2}, \quad b' \in \mathcal{A}_b, \quad b_1' \in \mathcal{A}_{b_1}, \quad b_2' \in \mathcal{A}_{b_2},
\]
and \( R_1', R_2' \) such that such that the final value of \( T \) is
\[
[a' \sqcap \text{gslice}(B, a, a_1, a_1') \sqcap \text{gslice}(\neg B, a, a_2, a_2')] [b'] \quad \text{if } B \text{ then } [a_1' \sqcap \text{btrack}(\gamma \circ a, b_1, b, b')] [b'],
\]
and \([a_1' \sqcap \text{gslice}(\gamma \circ a, b_1, b, b')] \) is the final value of \( T_1 \) for \( i = 1, 2 \). For each \( i = 1, 2 \), the value of \( T_i \) does not change after the recursive call for \( T_i \) in \( \text{gslice} \). Thus, by the induction
hypothesis, the final values of $T_1$ and $T_2$ satisfy all the constraints in $C([a_1]R[b_1]) \cup C([a_2]R_2[b_2])$. We now need to show that the following four constraints hold:

\[
\begin{align*}
[B](a' \triangledown \text{gbslice}(B, a, a_1, a'_1) \triangledown \text{gbslice}(\neg B, a, a_2, a'_2)) & \sqsubseteq a'_1 \\
\neg[B](a' \triangledown \text{gbslice}(B, a, a_1, a'_1) \triangledown \text{gbslice}(\neg B, a, a_2, a'_2)) & \sqsubseteq a'_2 \\
(\gamma \circ \alpha)(b'_1 \triangledown \text{btrack}(\gamma \circ \alpha, b_1, b')) & \sqsubseteq b \\
(\gamma \circ \alpha)(b'_2 \triangledown \text{btrack}(\gamma \circ \alpha, b_2, b')) & \sqsubseteq b
\end{align*}
\]

The first two hold because of the “soundness” of \text{gbslice} (Lemma 5), and the other two constraints hold by the assumption on \text{btrack}.

Case $A \equiv [a]([\text{inv } i]\text{while } B \text{ do } [a_1]R[b_1])[b]$: Let $T_1$ be a program annotated with variables such that $T = [X][[\text{inv } K]\text{while } B \text{ do } T_1][Y]$ for some variables $X, K, Y$. By the definition of \text{gslice}, there exist

\[
a' \in A_a, \ a'_1 \in A_{a_1}, \ i' \in A_i, \ b' \in A_b, \ b'_1 \in A_{b_1}, \text{ and } R'_1
\]

such that

1. the final value of $T$ is

\[
[a' \triangledown \text{btrack}(\gamma \circ \alpha, a, i)(i')][[\text{inv } i']\text{while } B \text{ do } [a'_1]R'_1[b'_1])[b'],
\]

2. $b'_1 = b'_1 \triangledown \text{btrack}(\gamma \circ \alpha, b_1, i)(i')$, and

3. $i' = i' \triangledown \text{gbslice}(\neg B, i, b, b') \triangledown \text{gbslice}(B, i, b_1, b'_1)$.

By the definition of \text{gslice}, the final value of $T_1$ must have been obtained by the recursive call to \text{gslice}. Thus, by the induction hypothesis, all the constraints in $C([a'_1]R_1[b'_1])$ hold. It remains to show that the following constraints also hold:

\[
[B]i' \sqsubseteq a'_1, \ \neg[B]i' \sqsubseteq b'. \ (\gamma \circ \alpha)(b'_1') \sqsubseteq i', \ (\gamma \circ \alpha)(a' \triangledown \text{btrack}(\gamma \circ \alpha, a, i)(i')) \sqsubseteq i'.
\]

The first two constraints hold, because \text{gbslice} is sound (Lemma 5), and $i' = i' \triangledown \text{gbslice}(\neg B, i, b, b') \triangledown \text{gbslice}(B, i, b_1, b'_1)$; and the other two constraints hold, because $b'_1 = b'_1 \triangledown \text{btrack}(\gamma \circ \alpha, b_1, i)(i')$, and \text{btrack} is “sound” by assumption:

\[
\forall i'' \in A_i. \ (\gamma \circ \alpha)(\text{btrack}(\gamma \circ \alpha, b_1, i)(i'')) \sqsubseteq i''.
\]

\]
however, different: the abstract-value slicer works only on the results of an abstract interpreter, while program slicing modifies a program.

Our work is also related to the proposals for combining forward and backward abstract interpretations [CC99, Mas01]. The abstract-value slicer works backward; it backtracks each command, and finds out what parts of the “pre-abstract-value” the abstract interpreter used to compute the “crucial” parts of the “post-abstract-value.” Thus, adding the abstract-value slicer as a post-processor to an abstract interpreter can be seen roughly as a forward analysis followed by a backward analysis. However, our goal is different from existing proposals. The proposals mainly concern about improving the accuracy of the analysis, or speeding up the analysis time, whereas our goal is to obtain exactly the abstract interpretation results that contribute to our verification proofs.

We currently plan to extend the abstract-value slicing algorithm in Section 3 to handle more language features such as procedures, so that we can use the slicer for more realistic programs. Another plan is to test the generality of the framework in Section 5 with specific abstract
interpreters, including those for non-relational domains and the octagon domain [Min01b].

References


A Constraints for the Results of an Abstract Interpreter

Pruning Algorithm $[B] : M \rightarrow M$

| $[E \leq E']a$ is given | $[a]a = a$ |
| $[B_0 \land B_1]a = [B_0]a \land [B_1]a$ | $[B_0 \lor B_1]a = [B_0]a \lor [B_1]a$ |

Constraint Generation Algorithm $C$

$C([a]:=E[b]) = ([x:=E]a \sqsubseteq b)$

$C([a]\text{skip}[b]) = \{a \sqsubseteq b\}$

$C([a](\text{if } B \text{ then } ([a_1]R_1[b_1]) \text{ else } ([a_2]R_2[b_2]) )[b])$

$= \{[B]a \sqsubseteq a_1, [\neg B]a \sqsubseteq a_2, (\gamma \circ a)(b_1) \sqsubseteq b, (\gamma \circ a)(b_2) \sqsubseteq b \} \cup C([a_1]R_1[b_1]) \cup C([a_2]R_2[b_2])$

$C([a](\text{inv } i)\text{while } B \text{ do } ([a_1]R[b_1])[b])$

$= \{(\gamma \circ a)(i) \sqsubseteq i, (\gamma \circ a)(b_1) \sqsubseteq i, [B][i] \sqsubseteq a_1, [\neg B][i] \sqsubseteq b \} \cup C([a_1]R[b_1])$

$C([a]([a_1]R_1[b_1]; [a_2]R_2[b_2])[b])$

$= \{a \sqsubseteq a_1, b_1 \sqsubseteq a_2, b_2 \sqsubseteq b \} \cup C([a_1]R_1[b_1]) \cup C([a_2]R_2[b_2])$