On Expressive Power of Second Order Propositional Program Logics*

(extended abstract of a short talk)

N.V. Shilov †   K. Yi

Research On Program Analysis System (ROPAS)
Department of Computer Science
Korean Advanced Institute of Science and Technology (KAIST)
Kusong-dong Yusong-gu 373-1
Taejon 305-701, Republic of Korea
fax: +82 042 869 8780
{shilov, kwang}@ropas.kaist.ac.kr

Abstract

We examine expressive power of second order propositional program logics: logic $2M$ of C. Stirling and a new Second Order Elementary Propositional Dynamic Logic (SOEPDL). We demonstrate that SOEPDL is more expressive than $2M$, and them both are more expressive than the propositional μ-Calculus of D. Kozen (μC). We give also an "external" characteristic of SOEPDL expressive power: SOEPDL is as expressive as Second order Logic of monadic Successors of M. Rabin (**S(n)S**-Logic) in spite of different semantics. Thus, SOEPDL seems to be the most expressive state-based propositional program logic. Finally we discuss decidability issues of SOEPDL: undecidability of SOEPDL in general, but non-elementary decidability on infinite trees in particular (vs. elementary decidability of μC in all models and on infinite trees). We also give a new game-theoretic proof that μC is more expressive than a very popular with model checking community state-based Computation Tree Logic (CTL).

*This work is supported by Creative Research Initiatives of the Korean Ministry of Science and Technology

†While on leave from A.P. Ershov Institute of Informatics Systems of Siberian Division of Russian Academy of Science, Novosibirsk, Russia

1 SOEPDL vs. Propositional Program Logics

The propositional μ-Calculus of D. Kozen (μC) [6, 7] is a powerful propositional program logic with fixpoints. In particular, a very popular with model checking community state-based temporal Computation Tree Logic (CTL) [4, 2, 3] is expressible in μC. We give a new proof that CTL is less expressive than μC:

Proposition 1

1. *No CTL formula can express an existence of a winning strategy in finite games.*
2. *There is μC formula which expresses an existence of a winning strategy in finite games.*

But in spite of expressive power of μC , there exist more expressive propositional program logics. In particular, μC is expressible in the second order state-based program logic $2M$ of C. Stirling [13] while $2M$ is not expressible in μC:

Proposition 2

1. *No μC formula can express commutativeness of computations in finite models.*
2. *There is $2M$ formula which expresses commutativeness of computations in models.*

We suggest a new Second Order Elementary Propositional Dynamic Logic (SOEPDL). The only difference between SOEPDL and $2M$ is interpretation of modalities $\Box/\Diamond$: in SOEPDL they mean "for every/some state" while in $2M$ they mean "for every/some reachable state". We demonstrate that $2M$ is expressible in SOEPDL while SOEPDL is not expressible in $2M$:

Proposition 3

1. *No $2M$ formula can express the weakest preconditions for inverse computations in finite models.*
2. *There is SOEPDL formula which expresses the weakest preconditions for inverse computations in models.*

Thus, SOEPDL can be characterized as the most expressive state-based propositional program logic:

Theorem 1 *$CTL < \mu C < 2M < SOEPDL$ where all expressibilities have linear complexity and all inexpressibilities can be justified in finite models.*

2 SOEPDL vs. S(n)S-Logic

The "internal" characteristic of the expressive power of SOEPDL in terms of propositional program logics correlates with an "external" characteristic in terms of Second order logic of monadic Successors of M. Rabin (**S(n)S**-Logic) [8, 9, 1]. A comparison of SOEPDL and **S(n)S**-Logic is not straightforward, since SOEPDL and **S(n)S**-Logic have different semantics: SOEPDL is based on indivisible states while **S(n)S**-Logic is based on states generated by multiple first order variables, action symbols in SOEPDL are interpreted as binary relations in Kripke structures while functional symbols in **S(n)S**-Logic are interpreted as monadic successors in Herbrand model (i.e., full infinite **n**-fold trees).

The first problem can be resolved by consideration of formulae of **S(n)S**-Logic with a single (at most) free first order variable. In this case states generated by multiple first order variables can be identified with indivisible states presented by values of this single variable.

The last problem can be resolved by explicit references to classes of models: Herbrand models and Kripke structures. We would like to remark that Herbrand models are a particular case of Kripke structures.

Proposition 4 *For every SOEPDL formula it is possible to construct in linear time a formula of* **S(n)S***-Logic with a single (at most) free first order variable such that both formulae are equivalent in Kripke structures.*

Proposition 5 *For every formula of* **S(n)S***-Logic with a single (at most) free first order variable it is possible to construct in linear time SOEPDL formula such that both formulae are equivalent in Kripke structures.*

Combining propositions 4 and 5 we get

Theorem 2 *Expressive powers of SOEPDL and formulae of* **S(n)S***-Logic with a single (at most) free first order variable are linear time equivalent in Kripke structures in general as well as in Herbrand models in particular.*

It is known that μC extended by a formula for commutativeness of computations is undecidable [11]. It is also known that **S(n)S**-Logic is non-elementary decidable in Herbrand models [8, 9, 1]. These facts together with theorems 1 and 2 imply the following

Corollary

1. *$2M$ and SOEPDL are undecidable.*
2. *$2M$ and SOEPDL are non-elementary decidable in Herbrand Models.*
3. *Lower bound for SOEPDL in Herbrand Models is non-elementary.*

We would like to remark also that in Herbrand models μC $\cong$ **S(n)S**-Logic [10] but this time an interpretation of **S(n)S**-Logic in μC has non-elementary complexity.

3 Conclusion

We have demonstrated that CTL $<$ μC $<$ $2M$ $<$ SOEPDL $\cong$ $\mathbf{S(n)S}$-Logic and that SOEPDL is non-elementary decidable on infinite trees while is undecidable in general case. It is also well-known that μC is exponentially decidable on infinite trees as well as in general case [5]. In contrast to μC and SOEPDL, decidability bounds for $2M$ on infinite trees are still an open question. From one side, $2M$ is closely related to Quantified Propositional (linear) Temporal Logic (QPTL), and non-elementary decidability for QPTL has been established [12]. Simultaneously, $2M$ is closely related to another second order propositional program logic – Second Order Propositional Dynamic Logic (of program schemata) (SOPDL), and exponential upper bound for SOPDL on infinite trees has been proved in [11]. Thus more research are required for accurate decidability bounds of $2M$ on infinite trees.

References

[1] Börger E., Grädel E., Gurevich Y. *The Classical Decision Problem.* Springer, 1997.

[2] Burch J.R., Clarke E.M., McMillan K.L., Dill D.L., Hwang L.J. *Symbolic Model Checking: 10^{20} states and beyond.* Information and Computation, v.98, n.2, 1992, p.142-170.

[3] Clarke E.M., Grumberg O., Peled D. Model Checking. MIT Press, 1999.

[4] Emerson E.A. *Temporal and Modal Logic. Handbook of Theoretical Computer Science*, v.B, Elsevier and The MIT Press, 1990, 995-1072.

[5] Emerson E.A., Jutla C.S. *The Complexity of Tree Automata and Logics of Programs.* SIAM J. Comput., v.29, n.1, 1999, p.132-158.

[6] Kozen D. *Results on the Propositional Mu-Calculus.* Theoretical Computer Science, v.27, n.3, 1983, p.333-354.

[7] Kozen D., Tiuryn J. *Logics of Programs.* Handbook of Theoretical Computer Science, v.B, Elsevier and The MIT Press, 1990, 789-840.

[8] Rabin M.O. *Decidability of second order theories and automata on infinite trees.* Trans. Amer. Math. Soc., v.141, 1969, p.1-35.

[9] Rabin M.O. *Decidable Theories.* in Handbook of Mathematical Logic, ed. Barwise J. and Keisler H.J., North-Holland Pub. Co., 1977, 595-630.

[10] Schlingloff H. *On expressive power of Modal Logic on Trees.* LNCS, v.620, 1992, p.441-450.

[11] Shilov N.V. *Program schemata vs. automata for decidability of program logics.* Theoretical Computer Science, v.175, n.1, 1997, p.15-27.

[12] Sistla A.P., Vardi M.Y., Wolper P. *The Complementation Problem for Büchi Automata with Applications to Temporal Logics.* Theoretical Computer Science, v. 49, 1987, p.217-237.

[13] Stirling C. *Games and Modal Mu-Calculus.* Lecture Notes in Computer Science, v.1055, 1996, p.298-312.