Mini-ML Expressions & Operational Semantics

Expressions $\quad e ::=$

$\underline{\text{z}} \mid \underline{\text{s}}\ e \mid (\underline{\text{case}}\ e_1\ \underline{\text{of}}\ \underline{\text{z}} \to e_2 \mid \underline{\text{s}}\ x \to e_3)$	*Natural numbers*
$\mid \langle e_1, e_2 \rangle \mid \underline{\text{fst}}\ e \mid \underline{\text{snd}}\ e$	*Pairs*
$\mid \underline{\text{lam}}\ x.\ e \mid e_1\ e_2$	*Functions*
$\mid \underline{\text{let}}\ \underline{\text{val}}\ x = e_1\ \underline{\text{in}}\ e_2$ $\mid \underline{\text{let}}\ \underline{\text{name}}\ x = e_1\ \underline{\text{in}}\ e_2$	*Definitions*
$\mid \underline{\text{fix}}\ x.\ e$	*Recursion*
$\mid x$	*Variables*

Evaluation judgement: $\quad e \hookrightarrow v$

$$\frac{}{\underline{\text{z}} \hookrightarrow \underline{\text{z}}}\ \textsf{ev\_z} \qquad \frac{e \hookrightarrow v}{\underline{\text{s}}\ e \hookrightarrow \underline{\text{s}}\ v}\ \textsf{ev\_s}$$

$$\frac{e_1 \hookrightarrow \underline{\text{z}} \qquad e_2 \hookrightarrow v}{(\underline{\text{case}}\ e_1\ \underline{\text{of}}\ \underline{\text{z}} \to e_2 \mid \underline{\text{s}}\ x \to e_3) \hookrightarrow v}\ \textsf{ev\_case\_z}$$

$$\frac{e_1 \hookrightarrow \underline{\text{s}}\ v_1' \qquad [v_1'/x]e_3 \hookrightarrow v}{(\underline{\text{case}}\ e_1\ \underline{\text{of}}\ \underline{\text{z}} \to e_2 \mid \underline{\text{s}}\ x \to e_3) \hookrightarrow v}\ \textsf{ev\_case\_s}$$

Mini-ML Operational Semantics contd.

$$\frac{e_1 \hookrightarrow v_1 \qquad e_2 \hookrightarrow v_2}{\langle e_1, e_2 \rangle \hookrightarrow \langle v_1, v_2 \rangle}\ \textsf{ev\_pair}$$

$$\frac{e \hookrightarrow \langle v_1, v_2 \rangle}{\underline{\mathrm{fst}}\ e \hookrightarrow v_1}\ \textsf{ev\_fst} \qquad\qquad \frac{e \hookrightarrow \langle v_1, v_2 \rangle}{\underline{\mathrm{snd}}\ e \hookrightarrow v_2}\ \textsf{ev\_snd}$$

$$\frac{}{\underline{\mathrm{lam}}\ x.\ e \hookrightarrow \underline{\mathrm{lam}}\ x.\ e}\ \textsf{ev\_lam}$$

$$\frac{e_1 \hookrightarrow \underline{\mathrm{lam}}\ x.\ e_1' \qquad e_2 \hookrightarrow v_2 \qquad [v_2/x]e_1' \hookrightarrow v}{e_1\ e_2 \hookrightarrow v}\ \textsf{ev\_app}$$

$$\frac{e_1 \hookrightarrow v_1 \qquad [v_1/x]e_2 \hookrightarrow v}{\underline{\mathrm{let}}\ \underline{\mathrm{val}}\ x = e_1\ \underline{\mathrm{in}}\ e_2 \hookrightarrow v}\ \textsf{ev\_letv}$$

$$\frac{[e_1/x]e_2 \hookrightarrow v}{\underline{\mathrm{let}}\ \underline{\mathrm{name}}\ x = e_1\ \underline{\mathrm{in}}\ e_2 \hookrightarrow v}\ \textsf{ev\_letn}$$

$$\frac{[\underline{\mathrm{fix}}\ x.\ e/x]e \hookrightarrow v}{\underline{\mathrm{fix}}\ x.\ e \hookrightarrow v}\ \textsf{ev\_fix}$$

Mini-ML Types & Typing Derivations

$$\text{Types} \quad \tau \;::=\; \mathbf{nat} \mid \tau_1 \times \tau_2 \mid \tau_1 \to \tau_2 \mid \alpha$$

$$\text{Contexts} \quad \Gamma \;::=\; \cdot \mid \Gamma, x{:}\tau$$

Typing judgement: $\Gamma \triangleright e : \tau$

$$\frac{}{\Gamma \triangleright \underline{\mathrm{z}} : \mathbf{nat}}\,\mathsf{tp\_z} \qquad \frac{\Gamma \triangleright e : \mathbf{nat}}{\Gamma \triangleright \underline{\mathrm{s}}\ e : \mathbf{nat}}\,\mathsf{tp\_s}$$

$$\frac{\Gamma \triangleright e_1 : \mathbf{nat} \qquad \Gamma \triangleright e_2 : \tau \qquad \Gamma, x{:}\mathbf{nat} \triangleright e_3 : \tau}{\Gamma \triangleright (\underline{\mathrm{case}}\ e_1\ \underline{\mathrm{of}}\ \underline{\mathrm{z}} \to e_2 \mid \underline{\mathrm{s}}\ x \to e_3) : \tau}\,\mathsf{tp\_case}$$

$$\frac{\Gamma \triangleright e_1 : \tau_1 \qquad \Gamma \triangleright e_2 : \tau_2}{\Gamma \triangleright \langle e_1, e_2 \rangle : \tau_1 \times \tau_2}\,\mathsf{tp\_pair}$$

$$\frac{\Gamma \triangleright e : \tau_1 \times \tau_2}{\Gamma \triangleright \underline{\mathrm{fst}}\ e : \tau_1}\,\mathsf{tp\_fst} \qquad \frac{\Gamma \triangleright e : \tau_1 \times \tau_2}{\Gamma \triangleright \underline{\mathrm{snd}}\ e : \tau_2}\,\mathsf{tp\_snd}$$

Mini-ML Typing Derivations contd.

$$\frac{\Gamma, x{:}\tau_1 \triangleright e : \tau_2}{\Gamma \triangleright \underline{\mathrm{lam}}\ x.\ e : \tau_1 \rightarrow \tau_2}\ \mathsf{tp\_lam}$$

$$\frac{\Gamma \triangleright e_1 : \tau_2 \rightarrow \tau_1 \qquad \Gamma \triangleright e_2 : \tau_2}{\Gamma \triangleright e_1\ e_2 : \tau_1}\ \mathsf{tp\_app}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \triangleright x : \tau}\ \mathsf{tp\_var}$$

$$\frac{\Gamma \triangleright e_1 : \tau_1 \qquad \Gamma, x{:}\tau \triangleright e_2 : \tau_2}{\Gamma \triangleright \underline{\mathrm{let}}\ \underline{\mathrm{val}}\ x = e_1\ \underline{\mathrm{in}}\ e_2 : \tau_2}\ \mathsf{tp\_letv}$$

$$\frac{\Gamma \triangleright e_1 : \tau_1 \qquad \Gamma \triangleright [e_1/x]e_2 : \tau_2}{\Gamma \triangleright \underline{\mathrm{let}}\ \underline{\mathrm{name}}\ x = e_1\ \underline{\mathrm{in}}\ e_2 : \tau_2}\ \mathsf{tp\_letn}$$

$$\frac{\Gamma, x{:}\tau \triangleright e : \tau}{\Gamma \triangleright \underline{\mathrm{fix}}\ x.\ e : \tau}\ \mathsf{tp\_fix}.$$

Type Preservation Lemmas

Lemma 2 (Inversion) *Given a context* Γ *and an expression* e *such that there exists a* τ *such that* $\Gamma \triangleright e : \tau$ *is derivable. Then the last inference rule of any derivation of* $\Gamma \triangleright e : \tau'$ *for some* τ' *is uniquely determined.*

Lemma 3 (Weakening and Exchange)

1. *If valid contexts* Γ *and* Γ' *only differ in the order of declarations, then* $\Gamma \triangleright e : \tau$ *if and only if* $\Gamma' \triangleright e : \tau$.
2. *If* $\Gamma \triangleright e : \tau$ *then* $\Gamma, \Gamma' \triangleright e : \tau$ *provided* Γ, Γ' *is a valid context.*

Lemma 4 (Substitution) *If* $\Gamma \triangleright e' : \tau'$ *and* $\Gamma, x{:}\tau' \triangleright e : \tau$, *then* $\Gamma \triangleright [e'/x]e : \tau$.

Type Preservation

Theorem 5 (Type Preservation) *For any e and v, if $e \hookrightarrow v$ is derivable, then for any τ such that $\triangleright\, e : \tau$ is derivable, $\triangleright\, v : \tau$ is also derivable.*

Proof: By induction on the structure of the deduction $\mathcal{D}$ of $e \hookrightarrow v$. QeD

Case: $\mathcal{D} =$

$$\dfrac{\begin{array}{ccc}\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ e_1 \hookrightarrow \underline{\mathrm{lam}}\ x.\ e_1' & e_2 \hookrightarrow v_2 & [v_2/x]e_1' \hookrightarrow v\end{array}}{e_1\ e_2 \hookrightarrow v}\ \mathsf{ev\_app}$$

$\triangleright\, e_1\ e_2 : \tau$	By assumption
$\triangleright\, e_1 : \tau_2 \to \tau$ and	
$\quad\triangleright\, e_2 : \tau_2$ for some τ_2	By inversion
$\triangleright\, \underline{\mathrm{lam}}\ x.\ e_1' : \tau_2 \to \tau$	By ind. hyp. on $\mathcal{D}_1$
$x{:}\tau_2 \triangleright\, e_1' : \tau$	By inversion
$\triangleright\, v_2 : \tau_2$	By ind. hyp. on $\mathcal{D}_2$
$\triangleright\, [v_2/x]e_1' : \tau$	By the Substitution Lemma
$\triangleright\, v : \tau$	By ind. hyp. on $\mathcal{D}_3$

LF Type Theory

Kinds	K	$::=$	$\mathrm{type} \mid \Pi x{:}A.\ K$
Families	A	$::=$	$a \mid A\ M \mid \Pi x{:}A_1.\ A_2$
Objects	M	$::=$	$c \mid x \mid \lambda x{:}A.\ M \mid M_1\ M_2$
Signatures	Σ	$::=$	$\cdot \mid \Sigma, a{:}K \mid \Sigma, c{:}A$
Contexts	Γ	$::=$	$\cdot \mid \Gamma, x{:}A$

$\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : K$	A is a valid family of kind K
$\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M : A$	M is a valid object of type A
$\Gamma \vdash^{\mathrm{LF}}_{\Sigma} K\ \mathit{Kind}$	K is a valid kind
$\vdash^{\mathrm{LF}} \Sigma\ \mathit{Sig}$	Σ is a valid signature
$\vdash^{\mathrm{LF}}_{\Sigma} \Gamma\ \mathit{Ctx}$	Γ is a valid context
$M \equiv N$	M is definitionally equal to N
$A \equiv B$	A is definitionally equal to B
$K \equiv K'$	K is definitionally equal to K'

Typing Derivations for LF

$$\frac{}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} \mathrm{type}\ \mathit{Kind}}\ \textsf{kndtyp}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : \mathrm{type} \qquad \Gamma, x{:}A \vdash^{\mathrm{LF}}_{\Sigma} K\ \mathit{Kind}}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} \Pi x{:}A.\ K\ \mathit{Kind}}\ \textsf{kndpi}$$

$$\frac{\Sigma(a) = K}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} a : K}\ \textsf{famcon}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : \mathrm{type} \qquad \Gamma, x{:}A \vdash^{\mathrm{LF}}_{\Sigma} B : \mathrm{type}}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} \Pi x{:}A.\ B : \mathrm{type}}\ \textsf{fampi}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : \Pi x{:}B.\ K \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} M : B}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A\ M : [M/x]K}\ \textsf{famapp}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : K \qquad K \equiv K' \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} K'\ \mathit{Kind}}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : K'}\ \textsf{kndcnv}$$

$$\frac{\Sigma(c) = A}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} c : A}\,\textsf{objcon} \qquad \frac{\Gamma(x) = A}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} x : A}\,\textsf{objvar}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : \mathrm{type} \qquad \Gamma, x{:}A \vdash^{\mathrm{LF}}_{\Sigma} M : B}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} \lambda x{:}A.\ M : \Pi x{:}A.\ B}\,\textsf{objlam}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M : \Pi x{:}A.\ B \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} N : A}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M\ N : [N/x]B}\,\textsf{objapp}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M : A \qquad A \equiv B \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} B : \mathrm{type}}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M : B}\,\textsf{typcnv}$$

$$\frac{}{\vdash^{\mathrm{LF}}_{\Sigma} \cdot\ \mathit{Ctx}}\,\textsf{ctxemp} \qquad \frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : \mathrm{type} \qquad \vdash^{\mathrm{LF}}_{\Sigma} \Gamma\ \mathit{Ctx}}{\vdash^{\mathrm{LF}}_{\Sigma} \Gamma, x{:}A\ \mathit{Ctx}}\,\textsf{ctxobj}$$

$$\frac{}{\vdash^{\mathrm{LF}} \cdot\ \mathit{Sig}}\,\textsf{sigemp}$$

$$\frac{\vdash^{\mathrm{LF}}_{\Sigma} K\ \mathit{Kind} \qquad \vdash^{\mathrm{LF}} \Sigma\ \mathit{Sig}}{\vdash^{\mathrm{LF}} \Sigma, a : K\ \mathit{valid}}\,\textsf{sigfam}$$

$$\frac{\vdash^{\mathrm{LF}}_{\Sigma} A : \mathrm{type} \qquad \vdash^{\mathrm{LF}} \Sigma\ \mathit{Sig}}{\vdash^{\mathrm{LF}} \Sigma, c : A\ \mathit{Sig}}\,\textsf{sigobj}$$

Conversions

$$\frac{}{(\lambda x{:}A.\ M)\ N \equiv [N/x]M}\ \textsf{beta}$$

$$\frac{}{(\lambda x{:}A.\ M\ x) \equiv M}\ \textsf{eta}$$

$$\frac{}{K \equiv K}\ \textsf{kndrefl} \qquad \frac{K' \equiv K}{K \equiv K'}\ \textsf{kndsym} \qquad \frac{K \equiv K'' \qquad K'' \equiv K'}{K \equiv K'}\ \textsf{kndtrans}$$

$$\frac{}{A \equiv A}\ \textsf{famrefl} \qquad \frac{B \equiv A}{A \equiv B}\ \textsf{famsym} \qquad \frac{A \equiv C \qquad C \equiv B}{A \equiv B}\ \textsf{famtrans}$$

$$\frac{}{M \equiv M}\ \textsf{objrefl} \qquad \frac{N \equiv M}{M \equiv N}\ \textsf{objsym} \qquad \frac{M \equiv O \qquad O \equiv N}{M \equiv N}\ \textsf{objtrans}$$

Congruence Rules

$$\frac{A \equiv A'}{\Pi x{:}A.\ K \equiv \Pi x{:}A'.\ K}\ \mathsf{cngkndpi}_1 \qquad \frac{K \equiv K'}{\Pi x{:}A.\ K \equiv \Pi x{:}A.\ K'}\ \mathsf{cngkndpi}_2$$

$$\frac{A \equiv A'}{A\ M \equiv A'\ M}\ \mathsf{cngfamapp}_1 \qquad \frac{M \equiv M'}{A\ M \equiv A\ M'}\ \mathsf{cngfamapp}_2$$

$$\frac{A_1 \equiv A_1'}{\Pi x{:}A_1.\ A_2 \equiv \Pi x{:}A_1'.\ A_2}\ \mathsf{cngfampi}_1 \qquad \frac{A_2 \equiv A_2'}{\Pi x{:}A_1.\ A_2 \equiv \Pi x{:}A_1.\ A_2'}\ \mathsf{cngfampi}_2$$

$$\frac{A \equiv A'}{\lambda x{:}A.\ M \equiv \lambda x{:}A'.\ M}\ \mathsf{cngobjlam}_1 \qquad \frac{M \equiv M'}{\lambda x{:}A.\ M \equiv \lambda x{:}A.\ M'}\ \mathsf{cngobjlam}_2$$

$$\frac{M_1 \equiv M_1'}{M_1\ M_2 \equiv M_1'\ M_2}\ \mathsf{cngobjapp}_1 \qquad \frac{M_2 \equiv M_2'}{M_1\ M_2 \equiv M_1\ M_2'}\ \mathsf{cngobjapp}_2$$

Canonical Forms for LF

$\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \Uparrow A$	M is canonical of type A
$\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \Uparrow \mathrm{type}$	A is a canonical type
$\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \downarrow A$	M is atomic of type A
$\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \downarrow K$	A is atomic of kind K

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \Uparrow \mathrm{type} \qquad \Gamma, x{:}A \vdash^{\mathrm{LF}}_{\Sigma} M \Uparrow B}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} \lambda x{:}A.\ M \Uparrow \Pi x{:}A.\ B}\ \mathsf{canpi}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \downarrow \mathrm{type} \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \downarrow A}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \Uparrow A}\ \mathsf{canatm}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \Uparrow A \qquad A \equiv B \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} B : \mathrm{type}}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \Uparrow B}\ \mathsf{cancnv}$$

$$\frac{\Sigma(c) = A}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} c \downarrow A}\,\mathsf{atmcon} \qquad\qquad \frac{\Gamma(x) = A}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} x \downarrow A}\,\mathsf{atmvar}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \downarrow \Pi x{:}A.\ B \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} N \Uparrow A}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M\ N \downarrow [N/x]B}\,\mathsf{atmapp}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \downarrow A \qquad A \equiv B \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} B : \mathrm{type}}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \downarrow B}\,\mathsf{atmcnv}$$

$$\frac{\Sigma(a) = K}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} a \downarrow K}\,\mathsf{attcon}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \downarrow \Pi x{:}B.\ K \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \Uparrow B}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A\ M \downarrow [M/x]K}\,\mathsf{attapp}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \downarrow K \qquad K \equiv K' \qquad \Gamma \vdash^{\mathrm{LF}}_{\Sigma} K'\ Kind}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \downarrow K'}\,\mathsf{attcnv}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \Uparrow \mathrm{type} \qquad \Gamma, x{:}A \vdash^{\mathrm{LF}}_{\Sigma} B \Uparrow \mathrm{type}}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} \Pi x{:}A.\ B \Uparrow \mathrm{type}}\,\mathsf{cntpi}$$

$$\frac{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \downarrow \mathrm{type}}{\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \Uparrow \mathrm{type}}\,\mathsf{cntatm}$$

Properties of LF

Assume Σ is a valid signature, and Γ a context valid in Σ. Then the following hold.

1. If $\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M \Uparrow A$ then $\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M : A$.

2. If $\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A \Uparrow \mathrm{type}$ then $\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : \mathrm{type}$.

3. For each object M such that $\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M : A$ there exists a unique object M' such that $M \equiv M'$ and $\Gamma \vdash^{\mathrm{LF}}_{\Sigma} M' \Uparrow A$. Moreover, M' can be effectively computed.

4. For each type A such that $\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A : \mathrm{type}$ there exists a unique type A' such that $A \equiv A'$ and $\Gamma \vdash^{\mathrm{LF}}_{\Sigma} A' \Uparrow \mathrm{type}$. Moreover, A' can be effectively computed.

5. Type checking in the LF type theory is decidable.

Signature for Mini-ML

$$
\begin{array}{lcl}
\mathrm{exp} & : & \mathrm{type} \\
\mathrm{z} & : & \mathrm{exp} \\
\mathrm{s} & : & \mathrm{exp} \rightarrow \mathrm{exp} \\
\mathrm{case} & : & \mathrm{exp} \rightarrow \mathrm{exp} \rightarrow (\mathrm{exp} \rightarrow \mathrm{exp}) \rightarrow \mathrm{exp} \\
\mathrm{pair} & : & \mathrm{exp} \rightarrow \mathrm{exp} \rightarrow \mathrm{exp} \\
\mathrm{fst} & : & \mathrm{exp} \rightarrow \mathrm{exp} \\
\mathrm{snd} & : & \mathrm{exp} \rightarrow \mathrm{exp} \\
\mathrm{lam} & : & (\mathrm{exp} \rightarrow \mathrm{exp}) \rightarrow \mathrm{exp} \\
\mathrm{app} & : & \mathrm{exp} \rightarrow \mathrm{exp} \rightarrow \mathrm{exp} \\
\mathrm{letv} & : & \mathrm{exp} \rightarrow (\mathrm{exp} \rightarrow \mathrm{exp}) \rightarrow \mathrm{exp} \\
\mathrm{letn} & : & \mathrm{exp} \rightarrow (\mathrm{exp} \rightarrow \mathrm{exp}) \rightarrow \mathrm{exp} \\
\mathrm{fix} & : & (\mathrm{exp} \rightarrow \mathrm{exp}) \rightarrow \mathrm{exp}
\end{array}
$$

Representation Function for Mini-ML

$$
\begin{array}{rcl}
\ulcorner \underline{\mathrm{z}} \urcorner & = & \mathrm{z} \\
\ulcorner \underline{\mathrm{s}}\ e \urcorner & = & \mathrm{s}\ \ulcorner e \urcorner \\
\ulcorner \underline{\mathrm{case}}\ e_1\ \underline{\mathrm{of}}\ \underline{\mathrm{z}} \rightarrow e_2 \mid \underline{\mathrm{s}}\ x \rightarrow e_3 \urcorner & = & \mathrm{case}\ \ulcorner e_1 \urcorner\ \ulcorner e_2 \urcorner\ (\lambda x{:}\mathrm{exp}.\ \ulcorner e_3 \urcorner) \\
\ulcorner \langle e_1, e_2 \rangle \urcorner & = & \mathrm{pair}\ \ulcorner e_1 \urcorner \ulcorner e_2 \urcorner \\
\ulcorner \underline{\mathrm{fst}}\ e \urcorner & = & \mathrm{fst}\ \ulcorner e \urcorner \\
\ulcorner \underline{\mathrm{snd}}\ e \urcorner & = & \mathrm{snd}\ \ulcorner e \urcorner \\
\ulcorner \underline{\mathrm{lam}}\ x.\ e \urcorner & = & \mathrm{lam}\ (\lambda x{:}\mathrm{exp}.\ \ulcorner e \urcorner) \\
\ulcorner e_1\ e_2 \urcorner & = & \mathrm{app}\ \ulcorner e_1 \urcorner\ \ulcorner e_2 \urcorner \\
\ulcorner \underline{\mathrm{let}}\ \underline{\mathrm{val}}\ x = e_1\ \underline{\mathrm{in}}\ e_2 \urcorner & = & \mathrm{letv}\ \ulcorner e_1 \urcorner\ (\lambda x{:}\mathrm{exp}.\ \ulcorner e_2 \urcorner) \\
\ulcorner \underline{\mathrm{let}}\ \underline{\mathrm{name}}\ x = e_1\ \underline{\mathrm{in}}\ e_2 \urcorner & = & \mathrm{letn}\ \ulcorner e_1 \urcorner\ (\lambda x{:}\mathrm{exp}.\ \ulcorner e_2 \urcorner) \\
\ulcorner \underline{\mathrm{fix}}\ x.\ e \urcorner & = & \mathrm{fix}\ (\lambda x{:}\mathrm{exp}.\ \ulcorner e \urcorner) \\
\ulcorner x \urcorner & = & x
\end{array}
$$

Lemma 6 (Representations are Canonical Forms) *For any context* $\Gamma = x_1{:}\mathrm{exp}, \ldots, x_n{:}\mathrm{exp}$ *and Mini-ML expression* e *with free variables among* $x_1, \ldots, x_n$*,*

$$\Gamma \vdash_E \ulcorner e \urcorner \Uparrow \mathrm{exp}$$

Inverse of Representation for Mini-ML

$$
\begin{array}{rcl}
\llcorner \mathrm{z} \lrcorner & = & \underline{\mathrm{z}} \\
\llcorner \mathrm{s}\ M \lrcorner & = & \underline{\mathrm{s}}\ \llcorner M \lrcorner \\
\llcorner \mathrm{case}\ M_1\ M_2\ (\lambda x{:}\mathrm{exp}.\ M_3) \lrcorner & = & \underline{\mathrm{case}}\ \llcorner M_1 \lrcorner\ \underline{\mathrm{of}}\ \underline{\mathrm{z}} \rightarrow \llcorner M_2 \lrcorner \mid \underline{\mathrm{s}}\ x \rightarrow \llcorner M_3 \lrcorner \\
\llcorner \mathrm{pair}\ M_1\ M_2 \lrcorner & = & \langle \llcorner M_1 \lrcorner, \llcorner M_2 \lrcorner \rangle \\
\llcorner \mathrm{fst}\ M \lrcorner & = & \underline{\mathrm{fst}}\ \llcorner M \lrcorner \\
\llcorner \mathrm{snd}\ M \lrcorner & = & \underline{\mathrm{snd}}\ \llcorner M \lrcorner \\
\llcorner \mathrm{lam}\ (\lambda x{:}\mathrm{exp}.\ M) \lrcorner & = & \underline{\mathrm{lam}}\ x.\ \llcorner M \lrcorner \\
\llcorner \mathrm{app}\ M_1\ M_2 \lrcorner & = & \llcorner M_1 \lrcorner\ \llcorner M_2 \lrcorner \\
\llcorner \mathrm{letv}\ M_1\ (\lambda x{:}\mathrm{exp}.\ M_2) \lrcorner & = & \underline{\mathrm{let}}\ \underline{\mathrm{val}}\ x = \llcorner M_1 \lrcorner\ \underline{\mathrm{in}}\ \llcorner M_2 \lrcorner \\
\llcorner \mathrm{letn}\ M_1\ (\lambda x{:}\mathrm{exp}.\ M_2) \lrcorner & = & \underline{\mathrm{let}}\ \underline{\mathrm{name}}\ x = \llcorner M_1 \lrcorner\ \underline{\mathrm{in}}\ \llcorner M_2 \lrcorner \\
\llcorner \mathrm{fix}\ (\lambda x{:}\mathrm{exp}.\ M) \lrcorner & = & \underline{\mathrm{fix}}\ x.\ \llcorner M \lrcorner \\
\llcorner x \lrcorner & = & x
\end{array}
$$

Lemma 7 (Canonical Forms are Representations)
For any $\Gamma = x_1{:}\mathrm{exp}, \ldots, x_n{:}\mathrm{exp}$ *and* M *such that* $\Gamma \vdash_E M \Uparrow \mathrm{exp}$,

$\llcorner M \lrcorner$ *is defined and yields a Mini-ML expression such that* $\ulcorner \llcorner M \lrcorner \urcorner = M$.

Adequacy of Mini-ML Representation

Lemma 8 *For any Mini-ML expression* e, $\llcorner\ulcorner e\urcorner\lrcorner = e$.

Lemma 9 (Substitution Lemma) $\ulcorner[e_1/x]e_2\urcorner = [\ulcorner e_1\urcorner/x]\ulcorner e_2\urcorner$.

Theorem 10 (Adequacy) *There* $\ulcorner\cdot\urcorner$ *is a compositional bijection between Mini-ML expressions with free variables among* $x_1, \ldots, x_n$ *and (canonical) LF objects* M *such that*

$$x_1{:}\mathrm{exp}, \ldots, x_n{:}\mathrm{exp} \vdash_E M \Uparrow \mathrm{exp}$$

is derivable. The bijection is compositional in the sense that

$$\ulcorner[e_1/x]e_2\urcorner = [\ulcorner e_1\urcorner/x]\ulcorner e_2\urcorner.$$

Representation of Evaluation

$$
\begin{array}{lll}
\text{eval} & : & \text{exp} \to \text{exp} \to \text{type} \\[2em]
\text{ev\_z} & : & \text{eval z z} \\[1em]
\text{ev\_s} & : & \Pi E{:}\text{exp}.\ \Pi V{:}\text{exp}.\ \text{eval}\ E\ V \to \text{eval}\ (\text{s}\ E)\ (\text{s}\ V) \\[1em]
\text{ev\_case\_z} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp}.\ \Pi E_3{:}\text{exp} \to \text{exp}.\ \Pi V{:}\text{exp}. \\
& & \quad \text{eval}\ E_1\ \text{z} \to \\
& & \quad \text{eval}\ E_2\ V \to \\
& & \quad \text{eval}\ (\text{case}\ E_1\ E_2\ E_3)\ V \\[1em]
\text{ev\_case\_s} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp}.\ \Pi E_3{:}\text{exp} \to \text{exp}.\ \Pi V_1'{:}\text{exp}.\ \Pi V{:}\text{exp}. \\
& & \quad \text{eval}\ E_1\ (\text{s}\ V_1') \to \\
& & \quad \text{eval}\ (E_3\ V_1')\ V \to \\
& & \quad \text{eval}\ (\text{case}\ E_1\ E_2\ E_3)\ V \\[1em]
\text{ev\_pair} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp}.\ \Pi V_1{:}\text{exp}.\ \Pi V_2{:}\text{exp}. \\
& & \quad \text{eval}\ E_1\ V_1 \to \\
& & \quad \text{eval}\ E_2\ V_2 \to \\
& & \quad \text{eval}\ (\text{pair}\ E_1\ E_2)\ (\text{pair}\ V_1\ V_2)
\end{array}
$$

$$\begin{array}{lll}
\text{ev_fst} & : & \Pi E{:}\text{exp}.\ \Pi V_1{:}\text{exp}.\ \Pi V_2{:}\text{exp}. \\
& & \quad \text{eval } E \ (\text{pair } V_1 \ V_2) \to \\
& & \quad \text{eval } (\text{fst} E) \ V_1
\end{array}$$

$$\begin{array}{lll}
\text{ev_snd} & : & \Pi E{:}\text{exp}.\ \Pi V_1{:}\text{exp}.\ \Pi V_2{:}\text{exp}. \\
& & \quad \text{eval } E \ (\text{pair } V_1 \ V_2) \to \\
& & \quad \text{eval } (\text{snd} E) \ V_2
\end{array}$$

$$\begin{array}{lll}
\text{ev_lam} & : & \Pi E{:}\text{exp} \to \text{exp}.\ \text{eval } (\text{lam } E) \ (\text{lam } E).
\end{array}$$

$$\begin{array}{lll}
\text{ev_app} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp}.\ \Pi E_1'{:}\text{exp} \to \text{exp}.\ \Pi V_2{:}\text{exp}.\ \Pi V{:}\text{exp}. \\
& & \quad \text{eval } E_1 \ (\text{lam } E_1') \to \\
& & \quad \text{eval } E_2 \ V_2 \to \\
& & \quad \text{eval } (E_1' \ V_2) \ V \to \\
& & \quad \text{eval } (\text{app } E_1 \ E_2) \ V
\end{array}$$

$$\begin{array}{lll}
\text{ev_letv} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp} \to \text{exp}.\ \Pi V_1{:}\text{exp}.\ \Pi V{:}\text{exp}. \\
& & \quad \text{eval } E_1 \ V_1 \to \\
& & \quad \text{eval } (E_2 \ V_1) \ V \to \\
& & \quad \text{eval } (\text{letv } E_1 \ E_2) \ V
\end{array}$$

$$\begin{array}{lll}
\text{ev_letn} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp} \to \text{exp}.\ \Pi V{:}\text{exp}. \\
& & \quad \text{eval } (E_2 \ E_1) \ V \to \\
& & \quad \text{eval } (\text{letv } E_1 \ E_2) \ V
\end{array}$$

$$\begin{array}{lll}
\text{ev_fix} & : & \Pi E{:}\text{exp} \to \text{exp}.\ \Pi V{:}\text{exp}. \\
& & \quad \text{eval } (E \ (\text{fix } E)) \ V \to \\
& & \quad \text{eval } (\text{fix } E) \ V
\end{array}$$

Representation of Types & Typing Derivations

$$
\begin{array}{lcl}
\mathrm{tp} & : & \mathrm{type} \\[1em]
\mathrm{nat} & : & \mathrm{tp} \\
\mathrm{cross} & : & \mathrm{tp} \to \mathrm{tp} \to \mathrm{tp} \\
\mathrm{arrow} & : & \mathrm{tp} \to \mathrm{tp} \to \mathrm{tp} \\[2em]
\mathrm{oftp} & : & \mathrm{exp} \to \mathrm{tp} \to \mathrm{type}
\end{array}
$$

$$
\begin{array}{lcl}
\mathrm{oftp\_z} & : & \mathrm{oftp}\ \mathrm{z}\ \mathrm{nat} \\[1em]
\mathrm{oftp\_s} & : & \Pi E{:}\mathrm{exp}.\ \mathrm{oftp}\ E\ \mathrm{nat} \to \mathrm{oftp}\ (\mathrm{s}\ E)\ \mathrm{nat} \\[1em]
\mathrm{oftp\_case} & : & \Pi E_1{:}\mathrm{exp}.\ \Pi E_2{:}\mathrm{exp}.\ \Pi E_3{:}\mathrm{exp} \to \mathrm{exp}.\ \Pi T{:}\mathrm{tp}. \\
& & \quad \mathrm{oftp}\ E_1\ \mathrm{nat} \to \\
& & \quad \mathrm{oftp}\ E_2\ T \to \\
& & \quad (\Pi x{:}\mathrm{exp}.\ \mathrm{oftp}\ x\ \mathrm{nat} \to \mathrm{oftp}\ (E_3\ x)\ T) \to \\
& & \quad \mathrm{oftp}\ (\mathrm{case}\ E_1\ E_2\ E_3)\ T
\end{array}
$$

$$
\begin{array}{lll}
\mathrm{oftp\_pair} & : & \Pi E_1{:}\mathrm{exp}.\ \Pi E_2{:}\mathrm{exp}.\ \Pi T_1{:}\mathrm{tp}.\ \Pi T_2{:}\mathrm{tp}. \\
 & & \quad \mathrm{oftp}\ E_1\ T_1 \to \\
 & & \quad \mathrm{oftp}\ E_2\ T_2 \to \\
 & & \quad \mathrm{oftp}\ (\mathrm{pair}\ E_1\ E_2)\ (\mathrm{cross}\ T_1\ T_2)
\end{array}
$$

$$
\begin{array}{lll}
\mathrm{oftp\_fst} & : & \Pi E{:}\mathrm{exp}.\ \Pi T_1{:}\mathrm{tp}.\ \Pi T_2{:}\mathrm{tp}. \\
 & & \quad \mathrm{oftp}\ E\ (\mathrm{cross}\ T_1\ T_2) \to \\
 & & \quad \mathrm{oftp}\ (\mathrm{fst}\ E)\ T_1
\end{array}
$$

$$
\begin{array}{lll}
\mathrm{oftp\_snd} & : & \Pi E{:}\mathrm{exp}.\ \Pi T_1{:}\mathrm{tp}.\ \Pi T_2{:}\mathrm{tp}. \\
 & & \quad \mathrm{oftp}\ E\ (\mathrm{cross}\ T_1\ T_2) \to \\
 & & \quad \mathrm{oftp}\ (\mathrm{fst}\ E)\ T_2
\end{array}
$$

$$
\begin{array}{lll}
\mathrm{oftp\_lam} & : & \Pi E{:}\mathrm{exp} \to \mathrm{exp}.\ \Pi T_1{:}\mathrm{tp}.\ \Pi T_2{:}\mathrm{tp}. \\
 & & \quad (\Pi x{:}\mathrm{exp}.\ \mathrm{oftp}\ x\ T_1 \to \mathrm{oftp}\ (E\ x)\ T_2) \to \\
 & & \quad \mathrm{oftp}\ (\mathrm{lam}\ E)\ (\mathrm{arrow}\ T_1\ T_2)
\end{array}
$$

$$
\begin{array}{lll}
\mathrm{oftp\_app} & : & \Pi E_1{:}\mathrm{exp}.\ \Pi E_2{:}\mathrm{exp}.\ \Pi T_1{:}\mathrm{tp}.\ \Pi T_2{:}\mathrm{tp}. \\
 & & \quad \mathrm{oftp}\ E_1\ (\mathrm{arrow}\ T_2\ T_1) \to \\
 & & \quad \mathrm{oftp}\ E_2\ T_2 \to \\
 & & \quad \mathrm{oftp}\ (\mathrm{app}\ E_1\ E_2)\ T_1
\end{array}
$$

$$
\begin{array}{lll}
\text{oftp\_letv} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp} \to \text{exp}.\ \Pi T_1{:}\text{tp}.\ \Pi T_2{:}\text{tp}. \\
& & \quad \text{oftp}\ E_1\ T_1 \to \\
& & \quad (\Pi x{:}\text{exp}.\ \text{oftp}\ x\ T_1 \to \text{oftp}\ (E_2\ x)\ T_2 \to \\
& & \quad \text{oftp}\ (\text{letv}\ E_1\ E_2)\ T_2
\end{array}
$$

$$
\begin{array}{lll}
\text{oftp\_letn} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp} \to \text{exp}.\ \Pi T_1{:}\text{tp}.\ \Pi T_2{:}\text{tp}. \\
& & \quad \text{oftp}\ E_1\ T_1 \to \\
& & \quad \text{oftp}\ (E_2\ E_1)\ T_2 \to \\
& & \quad \text{oftp}\ (\text{letv}\ E_1\ E_2)\ T_2
\end{array}
$$

$$
\begin{array}{lll}
\text{oftp\_fix} & : & \Pi E{:}\text{exp} \to \text{exp}.\ \Pi T{:}\text{tp}. \\
& & \quad (\Pi x{:}\text{exp}.\ \text{oftp}\ x\ T \to \text{oftp}\ (E\ x)\ V) \to \\
& & \quad \text{oftp}\ (\text{fix}\ E)\ T
\end{array}
$$

Representation of Type Preservation

$$
\begin{aligned}
\text{tps} \;:\; & \Pi E{:}\text{exp}.\ \Pi V{:}\text{exp}.\ \Pi T{:}\text{tp}.\\
& \text{eval}\ E\ V \to \text{oftp}\ E\ T \to \text{oftp}\ V\ T \to \text{type}
\end{aligned}
$$

$$
\begin{aligned}
\text{tps\_z} \;&:\; \text{tps z z nat ev\_z oftp\_z oftp\_z}\\[1em]
\text{tps\_s} \;&:\; \Pi E{:}\text{exp}.\ \Pi V{:}\text{exp}.\\
&\quad \Pi D{:}\text{eval}\ E\ V.\ \Pi P{:}\text{oftp}\ E\ \text{nat}.\ \Pi Q{:}\text{oftp}\ V\ \text{nat}.\\
&\qquad \text{tps}\ E\ V\ \text{nat}\ D\ P\ Q \to\\
&\qquad \text{tps}\ (\text{s}\ E)\ (\text{s}\ V)\ \text{nat}\\
&\qquad\quad (\text{ev\_s}\ E\ V\ D)\ (\text{oftp\_s}\ E\ P)\ (\text{oftp\_s}\ V\ Q)\\
&\vdots\\
\text{tps\_lam} \;&:\; \Pi E{:}(\text{arrow exp exp}).\ \Pi T_1{:}\text{tp}.\ \Pi T_2{:}\text{tp}.\\
&\quad \Pi P{:}(\Pi x{:}\text{exp}.\ \text{oftp}\ x\ T_1 \to \text{oftp}\ (E\ x)\ T_2).\\
&\qquad \text{tps}\ (\text{lam}\ E)\ (\text{lam}\ E)\ (\text{arrow}\ T_1\ T_2)\\
&\qquad\quad (\text{ev\_lam}\ E)\ (\text{oftp\_lam}\ E\ T_1\ T_2\ P)
\end{aligned}
$$

$$
\begin{array}{lll}
\text{tps\_app} & : & \Pi E_1{:}\text{exp}.\ \Pi E_2{:}\text{exp}.\ \Pi E_1'{:}\text{exp} \to \text{exp}. \\
& & \Pi V_2{:}\text{exp}.\ \Pi V{:}\text{exp}.\ \Pi T_1{:}\text{tp}.\ \Pi T_2{:}\text{tp}. \\
& & \Pi D_1{:}\text{eval}\ E_1\ (\text{lam}\ E_1').\ \Pi D_2{:}\text{eval}\ E_2\ V_2.\ \Pi D_3{:}\text{eval}\ (E_1'\ V_2)\ V. \\
& & \Pi P_1{:}\text{oftp}\ E_1\ (\text{arrow}\ T_1\ T_2).\ \Pi P_2{:}\text{oftp}\ E_2\ T_1. \\
& & \Pi Q_1{:}(\Pi x{:}\text{exp}.\ \text{oftp}\ x\ T1 \to \text{oftp}\ (E_1'\ x)\ T2). \\
& & \Pi Q_2{:}\text{oftp}\ V_2\ T_1.\ Q_3{:}\text{oftp}\ V\ T_2. \\
& & \quad \text{tps}\ E_1\ (\text{lam}\ E_1')\ (\text{arrow}\ T_1\ T_2)\ D_1\ P_1\ (\text{oftp\_lam}\ E_1'\ T_1\ T_2\ Q_1) \to \\
& & \quad \text{tps}\ E_2\ V_2\ T_1\ D_2\ P_2\ Q_2 \to \\
& & \quad \text{tps}\ (E_1'\ V_2)\ V\ T_2\ D_3\ (Q_1\ V_2\ Q_2)\ Q_3 \to \\
& & \quad \text{tps}\ (\text{app}\ E_1\ E_2)\ V\ T_2\ (\text{ev\_app}\ E_1\ E_2\ E_1'\ V_2\ V\ D_1\ D_2\ D_3) \\
& & \qquad (\text{oftp\_app}\ E_1\ E_2\ T_1\ T_2\ P_1\ P_2)\ Q_3
\end{array}
$$