Some References on Linear Logic

Samson Abramsky, Computational interpretations of linear logic, TCS 111:3 57, 1993.

G. Bierman, On intuitionistic linear logic,
TR 346, U. of Cambridge, Computer Laboratory, Aug. 1994.

Andrew Barber, Linear type theories, semantics and action calculi, PhD. thesis, Dept. of CS, U. of Edinburgh, 1997.

J.-Y. Girard, Linear logic, TCS, 50:1 102, 1987.

A. Scedrov, A brief guide to linear logic, Bulletin of the European Association for Theoretical Computer Science, v. 41, pp 154-165.

A. S. Troelstra, *Lectures on Linear Logic*,
CSLI Lecture Notes 29, Center for the Study of Language and Information, Standford California, 1992.

Philip Wadler, A taste of linear logic,
MFPS, Springer Verlag LNCS 711, Gdansk, Aug 1993.
(http://cm.bell-labs.com/cm/cs/who/wadler/
topics/linear-logic.html)

Linear LF

$$
\begin{array}{lrcl}
\text{Kinds} & K & ::= & \text{type} \mid \Pi x{:}A.\ K \\[1em]
\text{Families} & A & ::= & a \mid A\ M \mid \\
 & & & \Pi x{:}A_1.\ A_2 \mid \\
 & & & A_1 \multimap A_2 \mid \\
 & & & A_1 \mathbin{\&} A_2 \mid \top \\[1em]
\text{Objects} & M & ::= & c \mid x \mid \\
 & & & \lambda x{:}A.\ M \mid M_1\ M_2 \mid \\
 & & & \hat{\lambda} x{:}A.\ M \mid M_1 \,\hat{}\, M_2 \mid \\
 & & & \langle M_1, M_2 \rangle \mid \pi_1\ M \mid \pi_2\ M \mid \\
 & & & \langle\rangle \\[2em]
\text{Contexts} & \Gamma & ::= & \cdot \mid \Gamma, x{:}A \mid \Gamma, x\hat{:}A \\[1em]
\text{Signatures} & \Sigma & ::= & \cdot \mid \Sigma, a{:}K \mid \Sigma, c{:}A
\end{array}
$$

For more information see:

http://theory.stanford.edu/~iliano/LLF/LLF.html

Mini-ML with References

$$
\begin{array}{rrcl}
\textsf{Expressions} & e & ::= & \vdots \\
 & & & \mid x \\
 & & & \mid \underline{\mathrm{lam}}\ x.\ e \mid e_1\ e_2 \\
 & & & \vdots \\
 & & & \mid c \mid \mathrm{ref}\ e \mid !\ e \\
 & & & \mid e_1 := e_2 \mid e_1; e_2
\end{array}
$$

$$
\begin{array}{rrcl}
\textsf{Stores} & S & ::= & \cdot \mid S, c = v
\end{array}
$$

$$
\begin{array}{rrcl}
\textsf{Instructions} & i & ::= & \mathrm{ev}\ e \mid \mathrm{return}\ v \\
 & & & \vdots \\
 & & & \mid \mathrm{app}_1\ v\ e \mid \mathrm{ref}_1\ v \mid !_1\ v \mid v :=_1 e \mid v_1 :=_2 v_2
\end{array}
$$

$$
\begin{array}{rrcl}
\textsf{Continuations} & K & ::= & \mathrm{init} \mid K, \lambda x.\ i
\end{array}
$$

$$
\begin{array}{rrcl}
\textsf{Answers} & a & ::= & (S, v) \mid \mathrm{new}\ c.\ a
\end{array}
$$

MLR Evaluation

$$S \triangleright K \vdash i \hookrightarrow a$$

$$\frac{}{S \triangleright \mathrm{init} \vdash \mathrm{return}\ v \hookrightarrow (S, v)}\ \mathsf{st\_init}$$

$$\frac{S \triangleright K \vdash [v/x]i \hookrightarrow a}{S \triangleright (K, \lambda x.\ i) \vdash \mathrm{return}\ v \hookrightarrow a}\ \mathsf{st\_ret}$$

$$\frac{S \triangleright K \vdash \mathrm{return}\ \underline{\mathrm{lam}}\ x.\ e \hookrightarrow a}{S \triangleright K \vdash \mathrm{ev}\ \underline{\mathrm{lam}}\ x.\ e \hookrightarrow a}\ \mathsf{st\_lam}$$

$$\frac{S \triangleright (K, \lambda x.\ \mathrm{app}_1\ x\ e_2) \vdash \mathrm{ev}\ e_1 \hookrightarrow a}{S \triangleright K \vdash \mathrm{ev}\ e_1\ e_2 \hookrightarrow a}\ \mathsf{st\_app}$$

$$\frac{S \triangleright (K, \lambda x.\ [x/x_1]e_1) \vdash \mathrm{ev}\ e_2 \hookrightarrow a}{S \triangleright K \vdash \mathrm{app}_1\ (\underline{\mathrm{lam}}\ x_1.\ e_1)\ e_2 \hookrightarrow a}\ \mathsf{st\_app_1}$$

$$\frac{S \triangleright K \vdash \mathrm{return}\ c \hookrightarrow a}{S \triangleright K \vdash \mathrm{ev}\ c \hookrightarrow a}\ \mathsf{st\_cell}$$

$$\frac{S \triangleright (K, \lambda x.\ \mathrm{ref}_1\ x) \vdash \mathrm{ev}\ e \hookrightarrow a}{S \triangleright K \vdash \mathrm{ev}\ \mathrm{ref}\ e \hookrightarrow a}\ \mathsf{st\_ref}$$

$$\frac{(S, c = v) \triangleright K \vdash \mathrm{return}\ c \hookrightarrow a}{S \triangleright K \vdash \mathrm{ev}\ \mathrm{ref}_1\ v \hookrightarrow \mathrm{new}\ c.\ a}\ \mathsf{st\_ref}_1$$

$$\frac{S \triangleright (K, \lambda x.\ !_1\ x) \vdash \mathrm{ev}\ e \hookrightarrow a}{S \triangleright K \vdash \mathrm{ev}\ !\ e \hookrightarrow a}\ \mathsf{st\_!}$$

$$\frac{(S, c = v, S') \triangleright K \vdash \mathrm{return}\ v \hookrightarrow a}{(S, c = v, S') \triangleright K \vdash !_1\ c \hookrightarrow a}\ \mathsf{st\_!}_1$$

$$\frac{S \triangleright (K, \lambda x.\ \mathrm{ev}\ e_2) \vdash \mathrm{ev}\ e_1 \hookrightarrow a}{S \triangleright K \vdash \mathrm{ev}\ e_1; e_2 \hookrightarrow a}\ \mathsf{st\_;}$$

$$\frac{S \triangleright (K, \lambda x.\ x :=_1 e_2) \vdash \mathrm{ev}\ e_1 \hookrightarrow a}{S \triangleright K \vdash \mathrm{ev}\ e_1 := e_2 \hookrightarrow a}\ \mathsf{st\_} :=$$

$$\frac{S \triangleright (K, \lambda x.\ c :=_2 x) \vdash \mathrm{ev}\ e \hookrightarrow a}{S \triangleright K \vdash c :=_1 e \hookrightarrow a}\ \mathsf{st\_} :=_1$$

$$\frac{(S, c = v, S') \triangleright K \vdash \mathrm{return}\ \langle\rangle \hookrightarrow a}{(S, c = v', S') \triangleright K \vdash c :=_2 v \hookrightarrow a}\ \mathsf{st\_} :=_2$$

Pure Ordered Natural Deduction

Constrain hypotheses to be used in relative order. (neither exchange, nor weakening, nor contraction admissible)

Ordered hypothetical judgement:

$$\Omega \vdash A$$

$$\frac{}{z{:}A \vdash A}\ \mathbf{hyp}^z$$

Two implications: $\twoheadrightarrow, \rightarrowtail$

In Lambek calculus [Lambek 58]:

$$A \twoheadrightarrow B \;\equiv\; B/A$$
$$A \rightarrowtail B \;\equiv\; A\backslash B$$

$$\frac{\Omega\ z{:}A \vdash B}{\Omega \vdash A \twoheadrightarrow B} \twoheadrightarrow_I^z$$

$$\frac{\Omega \vdash A \twoheadrightarrow B \qquad \Omega_A \vdash A}{\Omega\ \Omega_A \vdash B} \twoheadrightarrow_E$$

$$\frac{z{:}A\ \Omega \vdash B}{\Omega \vdash A \rightarrowtail B} \rightarrowtail_I^z$$

$$\frac{\Omega \vdash A \rightarrowtail B \qquad \Omega_A \vdash A}{\Omega_A\ \Omega \vdash B} \rightarrowtail_E$$

Natural Deduction for Ordered Linear Logic

Combine previous three systems.

Compound hypothetical judgement with three contexts:

$$\Gamma; \Delta; \Omega \vdash A$$

(extends DILL [Barber 97])

Three hypothesis rules:

$$\frac{}{\Gamma_L\ x{:}A\ \Gamma_R; \cdot; \cdot \vdash A}\ \mathbf{hyp}^x$$

$$\frac{}{\Gamma; y{:}A; \cdot; \vdash A}\ \mathbf{hyp}^y$$

$$\frac{}{\Gamma; \cdot; z{:}A \vdash A}\ \mathbf{hyp}^z$$

Four implications: $\rightarrow, \multimap, \twoheadrightarrow, \rightarrowtail$

Unrestricted implications:

$$\frac{\Gamma\ x{:}A; \Delta; \Omega \vdash B}{\Gamma; \Delta; \Omega \vdash A \rightarrow B} \rightarrow_I^x$$

$$\frac{\Gamma; \Delta; \Omega \vdash A \rightarrow B \qquad \Gamma; \cdot; \cdot \vdash A}{\Gamma; \Delta; \Omega \vdash B} \rightarrow_E$$

Linear implication rules are similar.

Right ordered implications:

$$\frac{\Gamma; \Delta; \Omega\ z{:}A \vdash B}{\Gamma; \Delta; \Omega \vdash A \twoheadrightarrow B} \twoheadrightarrow_I^z$$

$$\frac{\Gamma; \Delta; \Omega \vdash A \twoheadrightarrow B \qquad \Gamma; \Delta_A; \Omega_A \vdash A}{\Gamma; \Delta \bowtie \Delta_A; \Omega\ \Omega_A \vdash B} \twoheadrightarrow_E$$

Left ordered implications are symmetric

Proof terms for OLL

x, y, z		variables
$\lambda x{:}A.\ M$	$M_1\, M_2$	unrestricted functions ($A \rightarrow B$)
$\hat{\lambda} y{:}A.\ M$	$M_1 \,\hat{}\, M_2$	linear functions ($A \multimap B$)
$\overset{>}{\lambda} z{:}A.\ M$	$M_1 {}^{>} M_2$	right-ordered functions ($A \twoheadrightarrow B$)
$\overset{<}{\lambda} z{:}A.\ M$	$M_1 {}^{<} M_2$	left-ordered functions ($A \rightarrowtail B$)

We use Church style terms to give each term a unique type.

A Curry style system with just x, $\lambda x.\ M$, $M_1\, M_2$ also possible.

Full Ordered Linear Logic

Intuitionistic linear logic + Lambek calculus

Implications:

$$\rightarrow \quad \multimap \quad \twoheadrightarrow \quad \rightarrowtail$$

Conjunctions:

$$\bullet \quad 1 \quad \& \quad \top$$

Disjunction:

$$\oplus \quad 0$$

Quantifiers:

$$\forall \quad \exists$$

Modalities:

$$! \quad ¡$$

Rules for modalities

$$\frac{\Gamma; \Delta; \cdot \vdash A}{\Gamma; \Delta; \cdot \vdash \text{¡}A}\ \text{¡}_I$$

$$\frac{\Gamma; \Delta; \Omega \vdash \text{¡}A \qquad \Gamma; \Delta_C\ A; \Omega_L\ \Omega_R \vdash C}{\Gamma; \Delta_C \bowtie \Delta; \Omega_L\ \Omega\ \Omega_R \vdash C}\ \text{¡}_E$$

$$\frac{\Gamma; \cdot; \cdot \vdash A}{\Gamma; \cdot; \cdot \vdash !A}\ !_I$$

$$\frac{\Gamma; \Delta; \Omega \vdash !A \qquad \Gamma\ A; \Delta_C; \Omega_L\ \Omega_R \vdash C}{\Gamma; \Delta_C \bowtie \Delta; \Omega_L\ \Omega\ \Omega_R \vdash C}\ !_E$$

Properties of OLL

Subject Reduction (for β reduction) [tlca99]:

If $M \Longrightarrow M'$ and $\Gamma; \Delta; \Omega \vdash M : A$ then
$\Gamma; \Delta; \Omega \vdash M' : A$.
(full normalisation also holds [mfps99])

also Subject Expansion (for η expansion)

Canonical Forms (long $\beta\eta$-normal form)
exist for implicational fragment [tlca99].

Proof by Kripke logical relations:

1. If M is in the logical relation represented by A then M has a canonical form N.
2. Build machinery of ordered substitutions
3. If M is a well-typed term of type A then M is in the logical relation represented by A.

DS Terms & CPS Terms

$$\begin{array}{lrcl}
\textit{DS Terms} & r & ::= & e \\
\textit{DS Expressions} & e & ::= & e_1\, e_2 \mid t \\
\textit{DS Trivial Expressions} & t & ::= & \lambda x.\, r \mid x
\end{array}$$

$$\begin{array}{lrcl}
\textit{Root Terms} & r & ::= & \lambda k.\, e \\
\textit{Serious Terms} & e & ::= & t_1\, t_2\, c \mid c\, t \\
\textit{Trivial Terms} & t & ::= & \lambda x.\, r \mid x \mid v \\
\textit{Continuation Terms} & c & ::= & \lambda v.\, e \mid k
\end{array}$$

Note that we are distinguishing variables x which are parameters of functions from variables v which are parameters to continuations.

CPS Transform
(left-to-right & call-by-value)

$$\vdash r \xrightarrow{DR} r' \qquad \vdash e \;;\; c \xrightarrow{DE} e' \qquad \vdash t \xrightarrow{DT} t'$$

$$\frac{\vdash e \;;\; k \xrightarrow{DE} e'}{\vdash e \xrightarrow{DR} \lambda k.\; e'} \qquad \frac{\vdash t \xrightarrow{DT} t'}{\vdash t \;;\; c \xrightarrow{DE} c\, t'}$$

$$\frac{\vdash e_2 \;;\; \lambda v_2.\; v_1\, v_2\, c \xrightarrow{DE} e_2' \qquad \vdash e_1 \;;\; \lambda v_1.\; e_2' \xrightarrow{DE} e'}{\vdash e_1\, e_2 \;;\; c \xrightarrow{DE} e'}$$

(v_1 not free in conclusion)

$$\frac{}{\vdash x \xrightarrow{DT} x} \qquad \frac{\vdash r \xrightarrow{DR} r'}{\vdash \lambda x.\; r \xrightarrow{DT} \lambda x.\; r'}$$

Ordering Constraints on CPS Terms

$$\models^{\mathbf{Root}} r \qquad \Phi \models^{\mathbf{Exp}} e \qquad \Phi \models^{\mathbf{Triv}} t; \Phi' \qquad \Phi \models^{\mathbf{Cont}} c$$

$$\Phi ::= \cdot \mid \Phi, k \mid \Phi, v$$

When Φ' is a prefix of Φ, we define $\Phi - \Phi'$ as the remainder of Φ.

$$\frac{k \models^{\mathbf{Exp}} e}{\models^{\mathbf{Root}} \lambda k.\, e}$$

$$\frac{\Phi \models^{\mathbf{Triv}} t; \Phi' \qquad \Phi' \models^{\mathbf{Cont}} c}{\Phi \models^{\mathbf{Exp}} c\, t} \qquad \frac{\Phi \models^{\mathbf{Triv}} t_1; \Phi' \quad \Phi' \models^{\mathbf{Triv}} t_0; \Phi'' \qquad \Phi'' \models^{\mathbf{Cont}} c}{\Phi \models^{\mathbf{Exp}} t_0\, t_1\, c}$$

$$\frac{}{\Phi \models^{\mathbf{Triv}} x; \Phi} \qquad \frac{\models^{\mathbf{Root}} r}{\Phi \models^{\mathbf{Triv}} \lambda x.\, r; \Phi} \qquad \frac{}{\Phi, v \models^{\mathbf{Triv}} v; \Phi}$$

$$\frac{}{k \models^{\mathbf{Cont}} k} \qquad \frac{\Phi, v \models^{\mathbf{Exp}} e}{\Phi \models^{\mathbf{Cont}} \lambda v.\, e}$$

Results of CPS transform satisfy ordering constraints

Would like to prove

$$\vdash r \xrightarrow{DR} r' \text{ implies } \models^{\mathbf{Root}} r$$

Not possible by simple structural induction.

Requires a "logical relations" style argument to correctly deal with expressions.

Olivier Danvy, Belmina Dzafic, and Frank Pfenning, On proving syntactic properties of CPS programs, HOOTS '99, Paris, France,Sept. 1999.

Olivier Danvy and Frank Pfenning, The occurrence of continuation parameters in CPS terms, TR CMU-CS-95-121, Dept. of CS, CMU, Feb 1995.

Possible to simplify proof by using an OLF representation.

DS Term Representation

$$\text{droot} : \text{type}. \qquad \text{dexp} : \text{type}. \qquad \text{dtriv} : \text{type}.$$

$$\begin{array}{rcl}
\text{e2r} & : & \text{dexp} \to \text{droot}. \\
\text{dapp} & : & \text{dexp} \to \text{dexp} \to \text{dexp}. \\
\text{t2e} & : & \text{dtriv} \to \text{dexp}. \\
\text{dlam} & : & (\text{triv} \to \text{droot}) \to \text{dtriv}.
\end{array}$$

$$\begin{array}{rcl}
\ulcorner e \urcorner^{R} & = & \text{e2r}\, \ulcorner e \urcorner^{E} \\
\\
\ulcorner e_0\, e_1 \urcorner^{E} & = & \text{dapp}\, \ulcorner e_0 \urcorner^{E}\, \ulcorner e_1 \urcorner^{E} \\
\ulcorner t \urcorner^{E} & = & \text{d2e}\, \ulcorner t \urcorner^{T} \\
\\
\ulcorner \lambda x.\, r \urcorner^{T} & = & \text{dlam}\, (\lambda x{:}\text{dtriv}.\ \ulcorner r \urcorner^{R}) \\
\ulcorner x \urcorner^{T} & = & x
\end{array}
\qquad
\begin{array}{rcl}
\llcorner \text{e2r}\, E \lrcorner_{R} & = & \llcorner E \lrcorner_{E} \\
\\
\llcorner \text{dapp}\, E_0\, E_1 \lrcorner_{E} & = & \llcorner E_0 \lrcorner_{E}\, \llcorner E_1 \lrcorner_{E} \\
\llcorner \text{d2e}\, T \lrcorner_{E} & = & \llcorner T \lrcorner_{T} \\
\\
\llcorner \text{dlam}\, (\lambda x{:}\text{dtriv}.\ R) \lrcorner_{T} & = & \lambda x.\, \llcorner R \lrcorner_{R} \\
\llcorner x \lrcorner_{T} & = & x
\end{array}$$

Adequacy easy to establish.

CPS Term Representation

root : type. exp : type. triv : type. cont : type.

$$
\begin{aligned}
\text{klam} &: (\text{cont} \twoheadrightarrow \text{exp}) \rightarrow \text{root}. \\
\text{kapp} &: \text{cont} \twoheadrightarrow \text{triv} \twoheadrightarrow \text{exp}. \\
\text{app} &: \text{cont} \twoheadrightarrow \text{triv} \twoheadrightarrow \text{triv} \twoheadrightarrow \text{exp}. \\
\text{lam} &: (\text{triv} \rightarrow \text{root}) \rightarrow \text{triv}. \\
\text{vlam} &: (\text{triv} \twoheadrightarrow \text{exp}) \twoheadrightarrow \text{cont}.
\end{aligned}
$$

Representation function for CPS terms

$$
\begin{array}{rcl@{\qquad}rcl}
\ulcorner \lambda k.\, e \urcorner & = & \mathrm{klam}\,(\overset{>}{\lambda k}{:}\mathrm{cont}.\ \ulcorner e \urcorner) & \llcorner \mathrm{klam}\,(\overset{>}{\lambda k}{:}\mathrm{cont}.\ E) \lrcorner & = & \lambda k.\, \llcorner E \lrcorner \\[1ex]
\ulcorner t_0\, t_1\, c \urcorner & = & \mathrm{app}^{>} \ulcorner c \urcorner^{>} \ulcorner t_0 \urcorner^{>} \ulcorner t_1 \urcorner & \llcorner \mathrm{app}^{>} C^{>} T_0^{>} T_1 \lrcorner & = & \llcorner T_0 \lrcorner \llcorner T_1 \lrcorner \llcorner C \lrcorner \\
\ulcorner c\, t \urcorner & = & \mathrm{kapp}^{>} \ulcorner c \urcorner^{>} \ulcorner t \urcorner & \llcorner \mathrm{kapp}^{>} C^{>} T \lrcorner & = & \llcorner C \lrcorner \llcorner T \lrcorner \\[1ex]
\ulcorner \lambda x.\, r \urcorner & = & \mathrm{lam}\,(\lambda x{:}\mathrm{triv}.\ \ulcorner r \urcorner) & \llcorner \mathrm{lam}\,(\lambda x{:}\mathrm{triv}.\ R) \lrcorner & = & \lambda x.\, \llcorner R \lrcorner \\
\ulcorner x \urcorner & = & x & \llcorner x \lrcorner & = & x \\
\ulcorner v \urcorner & = & v & \llcorner v \lrcorner & = & v \\[1ex]
\ulcorner \lambda v.\, e \urcorner & = & \mathrm{vlam}^{>}(\overset{>}{\lambda v}{:}\mathrm{triv}.\ \ulcorner e \urcorner) & \llcorner \mathrm{vlam}^{>}(\overset{>}{\lambda v}{:}\mathrm{triv}.\ E) \lrcorner & = & \lambda v.\, \llcorner E \lrcorner \\
\ulcorner k \urcorner & = & k & \llcorner k \lrcorner & = & k
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
\ulcorner \cdot \urcorner & = & \cdot & \llcorner \cdot \lrcorner & = & \cdot \\
\ulcorner \Phi, v \urcorner & = & \ulcorner \Phi \urcorner, v{:}\mathrm{triv} & \llcorner \Phi, v{:}\mathrm{triv} \lrcorner & = & \llcorner \Phi \lrcorner, v \\
\ulcorner \Phi, k \urcorner & = & \ulcorner \Phi \urcorner, k{:}\mathrm{cont} & \llcorner \Phi, k{:}\mathrm{cont} \lrcorner & = & \llcorner \Phi \lrcorner, k
\end{array}
$$

Note that and $\llcorner \ulcorner u \urcorner \lrcorner = u$ for any term u.

Theorem 1 (Representations are Canonical Forms)

Consider terms r, e, and t with free ordinary variables among $x_1, \ldots, x_n$ and let $\Gamma = x_1{:}\mathrm{triv} \ldots x_n{:}\mathrm{triv}$.

1. *If* $\models^{\mathbf{Root}} r$ *then* $\Gamma; \cdot \vdash \ulcorner r \urcorner \Uparrow \mathrm{root}$.

2. *If* $\Phi \models^{\mathbf{Exp}} e$ *then* $\Gamma; \ulcorner \Phi \urcorner \vdash \ulcorner e \urcorner \Uparrow \mathrm{exp}$.

3. *If* $\Phi \models^{\mathbf{Triv}} t; \Phi'$ *then* $\Gamma; \ulcorner (\Phi - \Phi') \urcorner \vdash \ulcorner t \urcorner \Uparrow \mathrm{triv}$.

4. *If* $\Phi \models^{\mathbf{Cont}} c$ *then* $\Gamma; \ulcorner \Phi \urcorner \vdash \ulcorner c \urcorner \Uparrow \mathrm{cont}$.

Proof by induction on the structure of the given derivations.

Theorem 2 (Canonical Forms are Representations)

Let $\Gamma = x_1{:}\mathrm{triv}, \ldots, x_n{:}\mathrm{triv}$ *be given.*

1. *For any* M *such that* $\Gamma; \cdot \vdash M \Uparrow \mathrm{root}$, $\llcorner M \lrcorner$ *is defined and* $\models^{\mathbf{Root}} \llcorner M \lrcorner$.

2. *For any* $\Omega = v_1{:}\mathrm{triv} \ldots v_n{:}\mathrm{triv}$ *and* M *with* $\Gamma; k{:}\mathrm{cont}, \Omega \vdash M \Uparrow \mathrm{exp}$, $\llcorner M \lrcorner$ *is defined and* $k, \llcorner \Omega \lrcorner \models^{\mathbf{Exp}} \llcorner M \lrcorner$.

3. *For any* $\Omega = v_1{:}\mathrm{triv} \ldots v_n{:}\mathrm{triv}$ *and* M *such that* $\Gamma; \Omega \vdash M \Uparrow \mathrm{triv}$, $\llcorner M \lrcorner$ *is defined and* $\Phi, \llcorner \Omega \lrcorner \models^{\mathbf{Triv}} \llcorner M \lrcorner; \Phi$ *for any* Φ.

4. *For any* $\Omega = v_1{:}\mathrm{triv} \ldots v_n{:}\mathrm{triv}$ *and* M *with* $\Gamma; k{:}\mathrm{cont}, \Omega \vdash M \Uparrow \mathrm{cont}$, $\llcorner M \lrcorner$ *is defined and* $k, \llcorner \Omega \lrcorner \models^{\mathbf{Cont}} \llcorner M \lrcorner$.

Proof by induction on the structure of the given canonical derivations.

Representation of CPS Transform

$$
\begin{aligned}
\text{cps\_r} &: \text{droot} \to \text{root} \to \text{type}.\\
\text{cps\_e} &: \text{dexp} \to \text{cont} \to \text{exp} \to \text{type}.\\
\text{cps\_t} &: \text{dtriv} \to \text{triv} \to \text{type}.
\end{aligned}
$$

$$
\begin{aligned}
\text{cps\_root} &: \Pi E{:}\text{dexp}.\ \Pi E'{:}\text{cont} \twoheadrightarrow \text{exp}.\\
&\quad (\Pi k{:}\text{cont}.\ \text{cps\_e}\, E\, k\, (E' \,{}^{>}\, k)) \to \text{cps\_r}\, (\text{e2r}\, E)\, (\text{klam}\, E').
\end{aligned}
$$

$$
\begin{aligned}
\text{cps\_app} &: \Pi E_0{:}\text{dexp}.\ \Pi E_1{:}\text{dexp}.\ \Pi C{:}\text{cont}.\ \Pi E_1'{:}\text{exp}.\ \Pi E'{:}\text{exp}.\\
&\quad \text{cps\_e}\, E_0\, (\text{vlam} \,{}^{>}\, E_1')\, E' \to\\
&\quad (\Pi v_0{:}\text{triv}.\ \text{cps\_e}\, E_1\, (\text{vlam} \,{}^{>}\, \overset{>}{\lambda} v_1{:}\text{triv}.\ \text{app} \,{}^{>}\, C \,{}^{>}\, v_0 \,{}^{>}\, v_1)\, (E_1' \,{}^{>}\, v_0)) \to\\
&\quad \text{cps\_e}\, (\text{dapp}\, E_0\, E_1)\, C\, E'.
\end{aligned}
$$

$$
\begin{aligned}
\text{cps\_triv} &: \Pi T{:}\text{dtriv}.\ \Pi C{:}\text{cont}.\ \Pi T'{:}\text{triv}.\\
&\quad \text{cps\_t}\, T\, T' \to \text{cps\_e}\, (\text{t2e}\, T)\, C\, (\text{kapp} \,{}^{>}\, C \,{}^{>}\, T').
\end{aligned}
$$

$$
\begin{aligned}
\text{cps\_lam} &: \Pi R{:}\text{droot}.\ \Pi R'{:}\text{root}.\\
&\quad (\Pi x{:}\text{dtriv}.\ \Pi x'{:}\text{triv}.\ \text{cps\_t}\, x\, x' \to \text{cps\_r}\, (R\, x)\, (R'\, x')) \to\\
&\quad \text{cps\_t}\, (\text{dlam}\, R)\, (\text{lam}\, R').
\end{aligned}
$$

Characterisation of contexts:

$$\begin{array}{rcl}
\Gamma & = & x_1\text{:dtriv} \ldots x_n\text{:dtriv} \\
\Gamma' & = & x'_1\text{:triv} \ldots x'_n\text{:triv} \\
\Gamma_m & = & m_1\text{:cps_t}\, x_1\, x'_1 \ldots m_n\text{:cps_t}\, x_n\, x'_n \\
\\
\Gamma_k & = & k_1\text{:cont} \ldots k_m\text{:cont} \\
\Gamma_v & = & v_1\text{:triv} \ldots v_l\text{:triv}
\end{array}$$

Theorem 3 (Representations are Canonical Forms) *Let $\Gamma^* = \Gamma, \Gamma', \Gamma_m, \Gamma_k, \Gamma_v$ be a context of the form explained above that contains all free variables occurring in the relevant judgment. Then*

1. $\vdash r \xrightarrow{DR} r'$ *implies*
   $\exists M.\ \ \Gamma^*; \cdot \vdash M \Uparrow \text{cps\_r} \ulcorner r \urcorner^R \ulcorner r' \urcorner.$
2. $\vdash e \;;\; c \xrightarrow{DE} e'$ *implies*
   $\exists M.\ \ \Gamma^*; \cdot; \cdot \vdash M \Uparrow \text{cps\_e} \ulcorner e \urcorner^E \ulcorner c \urcorner \ulcorner e' \urcorner$
3. $\vdash t \xrightarrow{DT} t'$ *implies*
   $\exists M.\ \ \Gamma^*; \cdot; \cdot \vdash M \Uparrow \text{cps\_t} \ulcorner t \urcorner^T \ulcorner t' \urcorner$

Proof by structural induction on the given derivation.

Theorem 4 (Canonical Forms are Representations) *Let* $\Gamma^* = \Gamma, \Gamma', \Gamma_m, \Gamma_k, \Gamma_v$ *a context of the form explained above and assume the types below are canonical.*

1. $\Gamma^*; \cdot; \cdot \vdash M \Uparrow \mathrm{cps\_r}\, R\, R'$ *implies* $\vdash \llcorner R \lrcorner_R \stackrel{DR}{\longrightarrow} \llcorner R' \lrcorner$.

2. $\Gamma^*; \cdot; \cdot \vdash M \Uparrow \mathrm{cps\_e}\, E\, C\, E'$ *implies* $\vdash \llcorner E \lrcorner_E\, ;\, \llcorner C \lrcorner \stackrel{DE}{\longrightarrow} \llcorner E' \lrcorner$.

3. $\Gamma^*; \cdot; \cdot \vdash M \Uparrow \mathrm{cps\_t}\, T\, T'$ *implies* $\vdash \llcorner T \lrcorner_T \stackrel{DR}{\longrightarrow} \llcorner T' \lrcorner$.

Proof by structural induction on the given canonical derivation.

Theorem 5 $\vdash r \stackrel{DR}{\longrightarrow} r'$ *implies* $\models^{\mathbf{Root}} r'$.

Proof: By theorem 3 we know $\cdot; \cdot; \cdot \vdash \ulcorner r' \urcorner \Uparrow \mathrm{root}$.
Then by theorem 2 we know $\models^{\mathbf{Root}} \llcorner \ulcorner r' \urcorner \lrcorner$.
Then we are done since $\llcorner \ulcorner r' \urcorner \lrcorner = r'$.