

SNU 4541.664A Program Analysis

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Note 3

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정리:공극을 드러내는 의미(*denotational semantics*)

- 모든 컴퓨터 프로그램의 의미(의미방정식의 해)는 의미공간 CPO의 한 원소이고, 그것은 의미방정식에 해당하는 연속함수의 최소 고정점(*least fixpoint*)으로 정의된다.
- 함수의 최소 고정점으로 정의된 프로그램의 의미에 대한 성질을 증명하는 데는 고정점 귀납법(*fixpoint induction*)이 유용하게 사용된다. 단, 그 성질이 품에 넣는 성질(*inclusive assertion*)일 때에만.

■ 프로그램 C 의 의미

$$[C] = \mathcal{F}([C]) \quad C \text{의 의미 방정식}$$

$$[C] \in D$$

$$\mathcal{F} \in D \rightarrow D$$

일 때,

$$[C] = \text{fix } \mathcal{F} \quad C \text{의 의미}$$

$$= \bigsqcup_{i \geq 0} (\mathcal{F}^i \perp)$$

■ 의미의 성질 $P([C])$ 을 증명하려면:

P 가 “품에 넣는 성질” 일 때

$$\forall x \in \{\perp, \mathcal{F}\perp, \mathcal{F}(\mathcal{F}\perp), \dots\}. P(x)$$

를 증명하면 됨

$$M \in \text{Memory} = \text{Var} \rightarrow \text{Value}$$

$$z \in \text{Value} = \mathbb{Z}_{\perp}$$

$$x \in \text{Var} = \text{Program Variable}_{\perp}$$

$$\llbracket C \rrbracket \in \text{Memory} \rightarrow \text{Memory}$$

$$\llbracket E \rrbracket \in \text{Memory} \rightarrow \text{Value}$$

$$\llbracket \text{skip} \rrbracket = \lambda M.M$$

$$\llbracket x := E \rrbracket = \lambda M.M\{x \mapsto \llbracket E \rrbracket M\}$$

$$\llbracket \text{if } E \text{ } C_1 \text{ } C_2 \rrbracket = \lambda M.\text{if } \llbracket E \rrbracket M \neq 0 \text{ then } \llbracket C_1 \rrbracket M \text{ else } \llbracket C_2 \rrbracket M$$

$$\llbracket C_1 ; C_2 \rrbracket = \lambda M.\llbracket C_2 \rrbracket (\llbracket C_1 \rrbracket M)$$

$$\llbracket \text{while } E \text{ do } C \rrbracket = \text{fix}(\lambda X.(\lambda M.\text{if } \llbracket E \rrbracket M \neq 0 \text{ then } X(\llbracket C \rrbracket M) \text{ else } M))$$

$$\llbracket n \rrbracket = \lambda M.n$$

$$\llbracket E_1 + E_2 \rrbracket = \lambda M.(\llbracket E_1 \rrbracket M) + (\llbracket E_2 \rrbracket M)$$

$$\llbracket - E \rrbracket M = \lambda M. - (\llbracket E \rrbracket M)$$

소품: 고정점(*fixpoint*) 예

“Computer science is full of fixpoints.”

귀납적/재귀적으로 표현되는 것 = 최소 고정점(*least fixpoint*)을 의미:

- $N = \{0\} \cup \{n+1 \mid n \in N\}$

$$N = \text{fix } \lambda X. \{0\} \cup \{n + 1 \mid n \in X\}$$

- $\text{list} = \{\text{nil}\} \cup \{(0, l) \mid l \in \text{list}\}$

$$\text{list} = \text{fix } \lambda X. \{\text{nil}\} \cup \{(0, l) \mid l \in X\}$$

$$\blacksquare \text{ reach}(N) = N \cup \text{reach}(\text{next}(N))$$

$$\text{reach} = \text{fix} \lambda f. (\lambda N. N \cup f(\text{next}(N)))$$

$$\blacksquare \text{ sort}(A) = \text{if sorted}(A)? A : \text{sort}(\text{exch}(A))$$

$$\text{sort} = \text{fix} \lambda f. (\lambda A. \text{if sorted}(A)? A : f(\text{exch}(A)))$$

- $\text{fac}(n) = \text{if } n=0? \ 1 : n*\text{fac}(n-1)$

$$\text{fac} = \text{fix } \lambda f. (\lambda n. \text{if } n = 0? \ 1 : n \times f(n - 1))$$

- $\text{repeat } C \ E = C ; \text{if } E \ (\text{repeat } C \ E) \ \text{skip}$

$$\llbracket \text{repeat } C \ E \rrbracket = \text{fix } \lambda X. ((\lambda M. \text{if } \llbracket E \rrbracket? \ X : \text{id}) \circ \llbracket C \rrbracket)$$

고정점 귀납법 (*fixpoint induction*) 예

$$\begin{aligned}
 & \llbracket \text{while } E \text{ do } C \rrbracket \\
 &= \text{fix } \lambda X. (\lambda M. \text{if } \llbracket E \rrbracket M \neq 0 \text{ then } X(\llbracket C \rrbracket M) \text{ else } M) \\
 & \llbracket \text{repeat } C \ E \rrbracket \\
 &= \text{fix } \lambda X. ((\lambda M. \text{if } \llbracket E \rrbracket? X : id) \circ \llbracket C \rrbracket)
 \end{aligned}$$

$$\text{증명: } \llbracket C ; \text{while } E \text{ do } C \rrbracket = \llbracket \text{repeat } C \ E \rrbracket$$

$$\llbracket C ; \text{while } E \text{ do } C \rrbracket = (\text{fix } F_w) \circ \llbracket C \rrbracket$$

$$\llbracket \text{repeat } C \text{ } E \rrbracket = \text{fix } F_r$$

$$F_w, F_r = \dots$$

증명할 것은

$$\forall C, E. (\text{fix } F_w) \circ \llbracket C \rrbracket = \text{fix } F_r$$

- 품에 넣는 성질 (*inclusive predicate*)? 예
- 어떤 고정점에 대한 성질? 연속함수 G 의 최소고정점에 대한:

$$G = \lambda \langle x_1, x_2 \rangle. \langle F_w x_1, F_r x_2 \rangle$$

왜냐면 $\text{fix } G = \langle \text{fix } F_w, \text{fix } F_r \rangle$

증명할것:

$$P(\text{fix } F_w, \text{fix } F_r) \stackrel{\text{let}}{=} (\forall C, E. (\text{fix } F_w) \circ \llbracket C \rrbracket = \text{fix } F_r)$$

고정점 귀납법(*fixpoint induction*)으로

- $P(\perp, \perp) \wedge P(F_w \perp, F_r \perp) \wedge P(F_w^2 \perp, F_w^2 \perp) \wedge \dots ?$ 즉,
- $P(\perp, \perp)$ 이고 $P(f, g) \Rightarrow P(F_w f, F_r g)$ 인가?