Engineering Mathematics II - Quiz 1 Solution

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1. (10pts) Inductively define the set of integers.

   Set of integers $\mathbb{Z}$ consists only of n that are inductively defined as follows:

$$n \;\rightarrow\; 0 \mid n-1 \mid n+1$$

□

2. (10pts) Inductively define the set of CNF formulars.

   Set of CNF formulars consists only of C that are inductively defined as follows:

$$\begin{array}{rcll} C & \rightarrow & F \mid C \wedge C & \\ F & \rightarrow & 0 \mid 1 & \\ & \mid & p & \text{variable} \\ & \mid & \neg p & \\ & \mid & F \vee F & \end{array}$$

□

3. (15pts) Show a natural deduction proof of $(p \rightarrow q) \wedge r \vdash p \rightarrow q$ using the sequent-style $(\Gamma \vdash \varphi)$ Natural Deduction proof rules.

   Proof)

$$\dfrac{\dfrac{}{(p \rightarrow q) \wedge r \vdash (p \rightarrow q) \wedge r}}{(p \rightarrow q) \wedge r \vdash p \rightarrow q}$$

□

4. (15pts) Show a natural deduction proof of

$$\vdash (p \wedge q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r) \wedge (q \rightarrow r)$$

   using the sequent-style $(\Gamma \vdash \varphi)$ Natural Deduction proof rules.

Proof)

let $G = (p \wedge q) \wedge (p \wedge q \rightarrow r)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{G, p \vdash G}}{G, p \vdash (p \wedge q)} \quad \dfrac{\dfrac{}{G, p \vdash G}}{G, p \vdash (p \wedge q) \rightarrow r}}{\dfrac{G, p \vdash r}{G \vdash p \rightarrow r}} \qquad \dfrac{\dfrac{\dfrac{}{G, q \vdash G}}{G, q \vdash (p \wedge q)} \quad \dfrac{\dfrac{}{G, q \vdash G}}{G, q \vdash (p \wedge q) \rightarrow r}}{\dfrac{G, q \vdash r}{G \vdash q \rightarrow r}}}{G \vdash (p \rightarrow r) \wedge (q \rightarrow r)}}{\vdash (p \wedge q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r) \wedge (q \rightarrow r)}$$

$\square$

5. (15pts) Show the steps that you check the validity of the following argument:

If

$$(1 \rightarrow x) \wedge (1 \rightarrow y) \wedge (x \wedge y \rightarrow p) \wedge (p \wedge z \rightarrow r)$$

is true, then

$$r$$

is true.

Proof)

$$(1 \rightarrow x) \wedge (1 \rightarrow y) \wedge (x \wedge y \rightarrow p) \wedge (p \wedge z \rightarrow r) \rightarrow r$$

위의 선언논리식이 참인지 보이기 위해서는 그것의 negation인

$$\neg((1 \rightarrow x) \wedge (1 \rightarrow y) \wedge (x \wedge y \rightarrow p) \wedge (p \wedge z \rightarrow r) \rightarrow r)$$

의 unsatisfiablity를 증명하면 된다. Horn Fomular로 변환하여 satisfiability를 점검해보자.

$$\begin{array}{rl}
(1 \rightarrow x) \wedge (1 \rightarrow y) \wedge (x \wedge y \rightarrow p) \wedge (p \wedge z \rightarrow r) \wedge (r \rightarrow 0) & \text{set } x = 1 \\
(1 \rightarrow y) \wedge (y \rightarrow p) \wedge (p \wedge z \rightarrow r) \wedge (r \rightarrow 0) & \text{set } y = 1 \\
(1 \rightarrow p) \wedge (p \wedge z \rightarrow r) \wedge (r \rightarrow 0) & \text{set } p = 1 \\
(z \rightarrow r) \wedge (r \rightarrow 0) & \text{satisfiable when } z = 0, r = 0
\end{array}$$

$\therefore$ not valid. $\square$

6. (15pts) Show the steps that you check the validity of the following argument:

If

$$q, r, s, v \rightarrow s, \text{ and } r \rightarrow p$$

are true, then

$$p, q, \text{ and } s$$

are true.

Proof) 5번 문제와 같은 방식으로 풀이한다.

$$\begin{array}{rl}
q, r, s, v \rightarrow s, r \rightarrow p, p \rightarrow 0, q \rightarrow 0, s \rightarrow 0 & \text{set } q, r, s = 1 \\
v \rightarrow 1, 1 \rightarrow p, p \rightarrow 0, 1 \rightarrow 0, 1 \rightarrow 0 & \text{set } v, p = 1 \\
1 \rightarrow 0, 1 \rightarrow 0, 1 \rightarrow 0 & \text{unsatisfiable}
\end{array}$$

$\therefore$ valid. $\square$

7. (2×5pts) O/X로 답하라.

- 별자리 운세가 항상 맞아떨어진다면, 별자리 운세법은 완전(complete)한 것이다 : X, 별자리 운세가 항상 맞아떨어진다면, 별자리 운세법은 안전(sound)한 것이다.
- 자연스런 유추법(Natural Deduction)이 유추해낼 수 없는 참인 선언논리식(propositional formula)는 없다 : O, 자연스런 유추법은 안전(sound)하고 완전(complete)하다. 따라서 모든 참인 선언논리식을 유추해낼 수 있으며 그 유추 결과는 항상 참이다. $\square$

8. Set $D \subseteq \mathbb{N} \times \mathbb{N}$ is inductively defined as follows:

   (a) for any $n \in \mathbb{N}$, (n, n) is an element of D.

   (b) for any $(n, m) \in D$, $(n, n \times m)$ is an element of D.

   (c) nothing else is an element of D.

   Prove:

   For any $(n, m) \in D$, there exists $i \in \mathbb{N}$ such that $m = n^i$.

   Proof) $\forall (n, m) \in D$에 대한 귀납법으로 증명한다.

   - `Base case` $m = n$: $n = n^1$이므로, $i = 1$인 i가 존재한다.
   - `Inductive case` $m' = n \times m$: 귀납가정에 의해 (n, m)에서 $m = n^i$이고 $m' = n^{i+1}$이므로, $i' = i + 1$인 i'이 존재한다.

   따라서, $\forall (n, m) \in D$에 대해 $m = n^i$인 자연수 i가 존재한다. $\square$