Engineering Mathematics II - Quiz 3 Solution

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1. (20pts=4pts/each) Translate the following English Statements into first-order predicate logic formula.

   - No number is the square root of 2.

   $$\forall x_{\in \mathbb{N}}.(\neg \texttt{Equal}(x, \sqrt{2}))$$

   - Every cubic polynomial with leading coefficient one has a root.

     x를 모든 cubic polynomial 집합의 한 원소라 할 때,

   $$\forall x.(\texttt{Equal}(\texttt{leadingCoefficient}(x), 1) \to \texttt{HasARoot}(x))$$

   - Every nonzero element has a multiplicative inverse.

   $$\forall x.(\neg \texttt{Equal}(x, 0) \to (\exists y.\texttt{Equal}(\texttt{multiply}(x, y), 1)))$$

   - The greatest common divisor of any two integers can be expressed as a linear combination of the two integers.

   $$\forall x_{\in \mathbb{Z}}.\forall y_{\in \mathbb{Z}}.\exists a_{\in \mathbb{Z}}.\exists b_{\in \mathbb{Z}}.\texttt{Equal}(\texttt{gcd}(x, y), \texttt{add}(\texttt{multiply}(a, x), \texttt{multiply}(b, y)))$$

   - If three points in a plane are not colinear, then there is a fourth point that is equidistant from all three.

   $$\forall x.\forall y.\forall z.(\neg \texttt{colinear}(x, y, z) \to \exists w.(\texttt{Equal}(\texttt{distance}(x, w), \texttt{distance}(y, w))$$

   $$\wedge \texttt{Equal}(\texttt{distance}(y, w), \texttt{distance}(z, w)))$$

   $\square$

2. (20pts) Prove the following by using the proof rules of the predicate logic:

   $$\vdash (\forall x.(A(x) \to B(x))) \to ((\exists x.A(x)) \to (\exists x.B(x)))$$

$G = \{\forall x.(A(x) \to B(x)), \exists x.A(x)\}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{G \cup \{A(y)\} \vdash A(y)} \quad \dfrac{\dfrac{}{G \cup \{A(y)\} \vdash \forall x.(A(x) \to B(x))}}{G \cup \{A(y)\} \vdash A(y) \to B(y)}}{G \cup \{A(y)\} \vdash B(y)}}{G \cup \{A(y)\} \vdash \exists x.B(x)}}{G \vdash \exists x.B(x)}}{\{\forall x.(A(x) \to B(x))\} \vdash (\exists x.A(x)) \to (\exists x.B(x))}}{\vdash (\forall x.(A(x) \to B(x))) \to ((\exists x.A(x)) \to (\exists x.B(x)))}
$$

$\square$

3. (20pts) Prove the validity of:

$$\exists x.(A(x) \to B(x)) \to ((\forall x.A(x)) \to (\exists x.B(x))) \qquad - \quad (1)$$

$\neg$ (1)이 unsatisfiable임을 보임으로써 (1)이 valid함을 보이겠다. 아래에 나오는 $\overset{skol}{\sim}$으로 연결되는 두식은 Scolemization에 의해 satisfiablity가 보존되는 관계이다.

$$
\begin{array}{lcll}
\exists x.(A(x) \to B(x)) & = & \exists x.(\neg A(x) \vee B(x)) & \\
& \overset{skol}{\sim} & \neg A(a) \vee B(a) & \text{by Skolemization} \\
\neg((\forall x.A(x)) \to (\exists x.B(x))) & = & \neg((\forall y.A(y)) \to (\exists z.B(z)) & \text{by renaming} \\
& = & \forall y.A(y) \wedge \neg(\exists z.B(z)) & \\
& = & \forall y.A(y) \wedge \forall z.\neg B(z) & \\
& = & \forall y.\forall z.(A(y) \wedge \neg B(z)) & \text{prenex form} \\
& \overset{skol}{\sim} & A(y) \wedge \neg B(z) & \text{by Skolemization}
\end{array}
$$

이제 resolution rule을 적용하자.

$$
\begin{array}{cl}
= & \{\neg A(a) \vee B(a), A(x), \neg B(y)\} \\
& \text{there are } \neg A(a) \vee B(a), A(x) \text{ and} \\
& \{a/x\} \text{ unifies } A(a), A(x) \text{ therefore add } B(a) \\
= & \{\neg A(a) \vee B(a), A(x), B(a), \neg B(y)\} \\
& \text{there are } B(a), \neg B(y) \text{ and} \\
& \{a/y\} \text{ unifies } B(a), B(y) \text{ therefore add } ;;
\end{array}
$$

따라서 (1)이 valid함을 보였다. $\square$

4. (40pts) Prove the validity of:

$$
\begin{array}{ll}
(& \forall x.(A(x,x)) \\
\wedge & \forall x.\forall y.(A(x,y) \to A(y,x)) \\
\wedge & \forall x.\forall y.\forall z.((A(x,y) \wedge A(y,z)) \to A(x,z)) \\
) & \to \forall x.\forall y.(\neg A(x,y) \to \forall z.(A(x,z) \to \neg A(y,z))) \qquad - \quad (2)
\end{array}
$$

$\neg$ (2)가 unsatisfiable 임을 보임으로써 (2)가 valid 함을 보이겠다.

$$
\begin{array}{lll}
\forall x.A(x,x) & \overset{skol}{\sim} & A(u,u) \\
 & & \text{by renaming and Skolemization} \\
\forall x.\forall y.(A(x,y) \to A(y,x)) & \overset{skol}{\sim} & A(v,w) \to A(w,v) \\
 & & \text{by renaming and Skolemization} \\
 & = & \neg A(v,w) \lor A(w,v) \\
\forall x.\forall y.\forall z.((A(x,y) \land A(y,z)) \to A(x,z)) & \overset{skol}{\sim} & (A(x,y) \land A(y,z)) \to A(x,z) \\
 & & \text{by Skolemization} \\
 & = & \neg A(x,y) \lor \neg A(y,z) \lor A(x,z) \\
\neg\forall x.\forall y.(\neg A(x,y) \to \forall z.(A(x,z) \to \neg A(y,z))) & = & \neg\forall x.\forall y.\forall z.(\neg A(x,y) \to (A(x,z) \to \neg A(y,z))) \\
 & & \text{prenex form} \\
 & = & \neg\forall x.\forall y.\forall z.(A(x,y) \lor (\neg A(x,z) \lor \neg A(y,z))) \\
 & = & \exists x.\exists y.\exists z.(\neg A(x,y) \land (A(x,z) \land A(y,z))) \\
 & \overset{skol}{\sim} & \neg A(a,b) \land A(a,c) \land A(b,c) \\
 & & \text{by Skolemization}
\end{array}
$$

이제 resolution rule을 적용하자.

$\{A(u,u), \neg A(v,w) \lor A(w,v), \neg A(x,y) \lor \neg A(y,z) \lor A(x,z), \neg A(a,b), A(a,c), A(b,c)\}$

there are $\neg A(x,y) \lor \neg A(y,z) \lor A(x,z), \neg A(a,b)$ and
$\{a/x, b/z\}$ unifies $A(x,z), A(a,b)$ therefore add $\neg A(a,y) \lor \neg A(y,b)$

$\{A(u,u), \neg A(v,w) \lor A(w,v), \neg A(x,y) \lor \neg A(y,z) \lor A(x,z), \neg A(a,b), A(a,c), A(b,c),$
$\neg A(a,y) \lor \neg A(y,b)\}$

there are $\neg A(v,w) \lor A(w,v), A(b,c)$ and
$\{b/v, c/w\}$ unifies $A(v,w), A(b,c)$ therefore add $A(c,b)$

$\{A(u,u), \neg A(v,w) \lor A(w,v), \neg A(x,y) \lor \neg A(y,z) \lor A(x,z), \neg A(a,b), A(a,c), A(b,c),$
$\neg A(a,y) \lor \neg A(y,b), A(c,b)\}$

there are $A(a,c), \neg A(a,y) \lor \neg A(y,b)$ and
$\{c/y\}$ unifies $A(a,c), A(a,y)$ therefore add $\neg A(c,b)$

$\{A(u,u), \neg A(v,w) \lor A(w,v), \neg A(x,y) \lor \neg A(y,z) \lor A(x,z), \neg A(a,b), A(a,c), A(b,c),$
$\neg A(a,y) \lor \neg A(y,b), A(c,b), \neg A(c,b)\}$

there are $A(c,b), \neg A(c,b)$ and
$\{\}$ unifies $A(c,b), A(c,b)$ therefore add ; ;

따라서 (2)가 valid함을 보였다. $\square$