Homework Set I Solution

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문제 1

E 에 대해 정리 1, C 에 대해 정리 2를 증명하고, 그 따름정리로써 주어진 명제가 성립함을 보인다.

정리 1.

$$\forall X, E, M, M'.\quad \mathrm{Var}(E) \subseteq X \text{이고 } M = M'|_X \text{이면}$$
$$[\![E]\!]\, M = [\![E]\!]\, M'$$

Proof. E 의 구조에 대한 귀납법으로 증명한다.

- $E = n$.

$$\begin{aligned}[\![n]\!]\, M &= n\\ &= [\![n]\!]\, M'\end{aligned}$$

- $E = x$.

$$\begin{aligned}[\![x]\!]\, M &= M(x)\\ &= M'(x) \quad (\because x \in \mathrm{Var}(E) \subseteq X \text{이고 } M = M'|_X)\\ &= [\![x]\!]\, M'\end{aligned}$$

- $E = E_1 \texttt{+} E_2$.
  $\mathrm{Var}(E_1) \subseteq \mathrm{Var}(E) \subseteq X$ 이므로 귀납가정에 의해 $[\![E_1]\!]\, M = [\![E_1]\!]\, M'$.
  마찬가지로 $[\![E_2]\!]\, M = [\![E_2]\!]\, M'$.

$$\begin{aligned}[\![E_1 \texttt{+} E_2]\!]\, M &= [\![E_1]\!]\, M + [\![E_2]\!]\, M\\ &= [\![E_1]\!]\, M' + [\![E_2]\!]\, M'\\ &= [\![E_1 \texttt{+} E_2]\!]\, M'\end{aligned}$$

- $E = \texttt{-}E_1$.
  앞의 경우에서와 같이 $[\![E_1]\!]\, M = [\![E_1]\!]\, M'$.

$$\begin{aligned}[\![\texttt{-}E_1]\!]\, M &= -\,[\![E_1]\!]\, M\\ &= -\,[\![E_1]\!]\, M'\\ &= [\![\texttt{-}E_1]\!]\, M'\end{aligned}$$

따라서 모든 E 에 대해 성립한다. □

정리 2.

$$\forall X, C, M, M'.\quad \mathrm{Var}(C) \subseteq X \text{이고 } M = M'|_X \text{이면}$$
$$\llbracket C \rrbracket\, M = \llbracket C \rrbracket\, M' = \bot$$
$$\text{또는}\quad \llbracket C \rrbracket\, M = \llbracket C \rrbracket\, M'|_X$$

Proof. C의 구조에 대한 귀납법으로 증명한다.

- $C = \texttt{skip}$.

$$\begin{aligned} \llbracket \texttt{skip} \rrbracket\, M &= M \\ &= M'|_X \quad (\because \text{가정}) \\ &= \llbracket \texttt{skip} \rrbracket\, M' \end{aligned}$$

- $C = x\ \texttt{:=}\ E$.
  $\mathrm{Var}(E) \subseteq \mathrm{Var}(C) \subseteq X$ 이므로 정리 1에 의해 $\llbracket E \rrbracket\, M = \llbracket E \rrbracket\, M'$.
  각각의 $y \in X$ 에 대해,

  - $y = x$ 인 경우,

$$\begin{aligned} (\llbracket x\ \texttt{:=}\ E \rrbracket\, M)(y) &= \llbracket E \rrbracket\, M \\ &= \llbracket E \rrbracket\, M' \\ &= (\llbracket x\ \texttt{:=}\ E \rrbracket\, M')(y). \end{aligned}$$

  - $y \neq x$ 인 경우,

$$\begin{aligned} (\llbracket x\ \texttt{:=}\ E \rrbracket\, M)(y) &= M(y) \\ &= M'(y) \quad (\because \text{가정}) \\ &= (\llbracket x\ \texttt{:=}\ E \rrbracket\, M')(y). \end{aligned}$$

  따라서 $\llbracket x\ \texttt{:=}\ E \rrbracket\, M = \llbracket x\ \texttt{:=}\ E \rrbracket\, M'|_X$.

- $C = C_1\texttt{;}\ C_2$.
  $\mathrm{Var}(C_1) \subseteq \mathrm{Var}(C) \subseteq X$ 이므로 귀납가정에 의해 아래 둘 중 하나가 성립한다.

  - $\llbracket C_1 \rrbracket\, M = \llbracket C_1 \rrbracket\, M' = \bot$

$$\llbracket C_2 \rrbracket\, (\llbracket C_1 \rrbracket\, M) = \llbracket C_2 \rrbracket\, (\llbracket C_1 \rrbracket\, M') = \llbracket C_2 \rrbracket\, (\bot) = \bot$$

  - $\llbracket C_1 \rrbracket\, M = \llbracket C_1 \rrbracket\, M'|_X$
    $\mathrm{Var}(C_2) \subseteq \mathrm{Var}(C) \subseteq X$ 이므로 귀납가정에 의해

$$\llbracket C_2 \rrbracket\, (\llbracket C_1 \rrbracket\, M) = \llbracket C_2 \rrbracket\, (\llbracket C_1 \rrbracket\, M') = \bot$$
$$\text{또는}\quad \llbracket C_2 \rrbracket\, (\llbracket C_1 \rrbracket\, M) = \llbracket C_2 \rrbracket\, (\llbracket C_1 \rrbracket\, M')|_X$$

  따라서 $\llbracket C_1\texttt{;}\ C_2 \rrbracket\, M = \llbracket C_1\texttt{;}\ C_2 \rrbracket\, M' = \bot$ 또는 $\llbracket C_1\texttt{;}\ C_2 \rrbracket\, M = \llbracket C_1\texttt{;}\ C_2 \rrbracket\, M'|_X$.

- $C = \texttt{if}\, E\, C_1\, C_2$.
  $\mathrm{Var}(E) \subseteq \mathrm{Var}(C) \subseteq X$ 이므로 정리 1에 의해 $\llbracket E \rrbracket\, M = \llbracket E \rrbracket\, M'$.

- $[\![E]\!]\, M = [\![E]\!]\, M' \neq 0$ 인 경우 $[\![\texttt{if}\ E\, C_1\, C_2]\!] = [\![C_1]\!]$.
  $\mathrm{Var}(C_1) \subseteq \mathrm{Var}(C) \subseteq X$ 이므로 귀납가정에 의해

$$\begin{aligned} & [\![C_1]\!]\, M = [\![C_1]\!]\, M' = \bot \\ \text{또는} \quad & [\![C_1]\!]\, M = [\![C_1]\!]\, M'|_X \end{aligned}$$

- $[\![E]\!]\, M = [\![E]\!]\, M' = 0$ 인 경우에도 위의 경우와 마찬가지로 성립한다.

• $C = \texttt{while}\ \ E\, C_1$.
  P, F 를 다음과 같이 정의하자.

$$\begin{aligned} F &\overset{\text{let}}{=} \lambda f.\lambda M.\ \text{if}\ [\![E]\!]\, M \neq 0\ \text{then}\ f([\![C_1]\!]\, M)\ \text{else}\ M \\ P(f) &\overset{\text{let}}{\Leftrightarrow} \forall M, M'.\ M = M'|_X \text{ 이면} \\ &\qquad\qquad f(M) = f(M') = \bot \text{ 또는 } f(M) = f(M')|_X \end{aligned}$$

그러면 $[\![\texttt{while}\ \ E\, C_1]\!] = \mathrm{lfp}(F)$ 이므로 $P(\mathrm{lfp}(F))$ 를 보이면 된다.

우선, P 가 품에 넣는 성질임을 보인다. 어떤 체인 C 에 대해 $\forall f \in C.P(f)$ 라고 가정하자. $M = M'|_X$ 이면 모든 $f \in C, x \in X$ 에 대해 $f(M)(x) = f(M')(x)$ 이므로,

$$\begin{aligned} (\sqcup C)(M)(x) &= \bigsqcup_{f \in C} f(M)(x) \\ &= \bigsqcup_{f \in C} f(M')(x) \\ &= (\sqcup C)(M')(x). \end{aligned}$$

즉 $P(\sqcup C)$ 가 성립하므로 P 는 품에 넣는 성질이다.

나머지 증명은 고정점 귀납법으로 진행한다.

- 바닥 증명:
  임의의 M, M' 에 대해 $\bot M = \bot = \bot M'$ 이므로 $P(\bot)$ 이 성립한다.
- 귀납 증명:
  $P(f)$ 가 성립한다고 가정하자. M, M' 에 대해 $M = M'|_X$ 라고 할 때, 정리 1에 의해 $[\![E]\!]\, M = [\![E]\!]\, M'$ 이다.
  * $[\![E]\!]\, M = [\![E]\!]\, M' \neq 0$ 인 경우 $F f M = f([\![C_1]\!]\, M), F f M' = f([\![C_1]\!]\, M')$.
    $\mathrm{Var}(C_1) \subseteq \mathrm{Var}(C) \subseteq X$ 이므로 귀납가정에 의해 아래 둘 중 하나가 성립한다.
    · $[\![C_1]\!]\, M = [\![C_1]\!]\, M' = \bot$

$$f([\![C_1]\!]\, M) = f([\![C_1]\!]\, M') = f(\bot) = \bot$$

    · $[\![C_1]\!]\, M = [\![C_1]\!]\, M'|_X$
      귀납 가정 $P(f)$ 에 의해

$$\begin{aligned} & f([\![C_1]\!]\, M) = f([\![C_1]\!]\, M') = \bot \\ \text{또는} \quad & f([\![C_1]\!]\, M) = f([\![C_1]\!]\, M')|_X \end{aligned}$$

* $[\![E]\!]\,M = [\![E]\!]\,M' = 0$ 인 경우,

$$
\begin{aligned}
FfM &= M \\
&= M'|_X \quad (\because \text{가정}) \\
&= FfM'
\end{aligned}
$$

P 가 품에 넣는 성질이고, $P(\bot)$ 가 성립하며, $P(f) \Rightarrow P(Ff)$ 이므로 고정점 귀납법에 의해 $P(\mathrm{lfp}(F))$ 가 성립한다.

□

따름정리 1.

$$
\begin{aligned}
\forall C, M, M'. \quad M = M'|_{\mathrm{Var}(C)} \text{이면} \\
[\![C]\!]\,M = [\![C]\!]\,M' = \bot \\
\text{또는} \quad [\![C]\!]\,M = [\![C]\!]\,M'|_{\mathrm{Var}(C)}
\end{aligned}
$$

Proof. $X = \mathrm{Var}(C)$ 로 놓으면 $\mathrm{Var}(C) \subseteq X$ 이므로 정리 2에 의해 성립한다. □

문제 2

(a) 증명나무

- `(fn x (x+1)) 2`

$$
\dfrac{\dfrac{}{\{\} \vdash \texttt{fn x (x+1)} \Downarrow (\texttt{fn x (x+1)}, \{\})} \quad \dfrac{}{\{\} \vdash 2 \Downarrow 2} \quad \dfrac{\dfrac{}{\{\texttt{x} \mapsto 2\} \vdash \texttt{x} \Downarrow 2} \quad \dfrac{}{\{\texttt{x} \mapsto 2\} \vdash 1 \Downarrow 1}}{\{\texttt{x} \mapsto 2\} \vdash \texttt{x+1} \Downarrow 3}}{\{\} \vdash \texttt{(fn x (x+1)) 2} \Downarrow 3}
$$

- `(fn x (x 1))(fn y (y+2))`

  $f_1 \overset{\text{let}}{=} (\texttt{fn x (x 1)}, \{\})$, $f_2 \overset{\text{let}}{=} (\texttt{fn y (y+2)}, \{\})$ 라고 하자.

$$
\dfrac{\dfrac{}{\{\} \vdash \texttt{fn x (x 1)} \Downarrow f_1} \quad \dfrac{}{\{\} \vdash \texttt{fn y (y+2)} \Downarrow f_2} \quad \dfrac{\dfrac{}{\{\texttt{x} \mapsto f_2\} \vdash \texttt{x} \Downarrow f_2} \quad \dfrac{}{\{\texttt{x} \mapsto f_2\} \vdash 1 \Downarrow 1} \quad \dfrac{\dfrac{}{\{\texttt{y} \mapsto 1\} \vdash \texttt{y} \Downarrow 1} \quad \dfrac{}{\{\texttt{y} \mapsto 1\} \vdash 2 \Downarrow 2}}{\{\texttt{y} \mapsto 1\} \vdash \texttt{y+2} \Downarrow 3}}{\{\texttt{x} \mapsto f_2\} \vdash \texttt{x 1} \Downarrow 3}}{\{\} \vdash \texttt{(fn x (x 1))(fn y (y+2))} \Downarrow 3}
$$

- `(rec f (fn x (if x 1 f(x-1)))) 1`

  $f \overset{\text{let}}{=} \mu\alpha.(\texttt{fn x (if x 1 f(x-1))}, \{\texttt{f} \mapsto \alpha\})$ 라고 하자. 전체 식의 실행의미는 다음과 같다.

$$
\dfrac{\dfrac{\dfrac{}{\{\texttt{f} \mapsto f\} \vdash \texttt{(fn x (if x 1 f(x-1)))} \Downarrow f}}{\{\} \vdash \texttt{rec f (fn x (if x 1 f(x-1)))} \Downarrow f} \quad \dfrac{}{\{\} \vdash 1 \Downarrow 1} \quad \dfrac{\dfrac{}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 1\} \vdash \texttt{x} \Downarrow 1} \quad \dfrac{(\ldots)}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 1\} \vdash \texttt{f(x-1)} \Downarrow 1}}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 1\} \vdash \texttt{if x 1 f(x-1)} \Downarrow 1}}{\{\} \vdash \texttt{(rec f (fn x (if x 1 f(x-1)))) 1} \Downarrow 1}
$$

  여기서 (...) 부분은 다음과 같다.

$$
\dfrac{\dfrac{}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 1\} \vdash \texttt{f} \Downarrow f} \quad \dfrac{\dfrac{}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 1\} \vdash \texttt{x} \Downarrow 1} \quad \dfrac{}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 1\} \vdash 1 \Downarrow 1}}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 1\} \vdash \texttt{x-1} \Downarrow 0} \quad \dfrac{\dfrac{}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 0\} \vdash \texttt{x} \Downarrow 0} \quad \dfrac{}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 0\} \vdash 1 \Downarrow 1}}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 0\} \vdash \texttt{if x 1 f(x-1)} \Downarrow 1}}{\{\texttt{f} \mapsto f, \texttt{x} \mapsto 1\} \vdash \texttt{f(x-1)} \Downarrow 1}
$$

(b) 작은보폭 실행발자국

- `(fn x (x+1)) 2`

$$
\begin{aligned}
&\texttt{(fn x (x+1)) 2}\\
&\rightarrow \{\texttt{2}/\texttt{x}\}\texttt{(x+1)}\\
&\ \ = \texttt{2+1}\\
&\rightarrow \texttt{3}
\end{aligned}
$$

- `(fn x (x 1))(fn y (y+2))`

$$
\begin{aligned}
&\texttt{(fn x (x 1))(fn y (y+2))}\\
&\rightarrow \{\texttt{(fn y (y+2))}/\texttt{x}\}\texttt{(x 1)}\\
&\ \ = \texttt{(fn y (y+2)) 1}\\
&\rightarrow \{\texttt{1}/\texttt{y}\}\texttt{(y+2)}\\
&\ \ = \texttt{1+2}\\
&\rightarrow \texttt{3}
\end{aligned}
$$

- `(rec f (fn x (if x 1 f(x-1)))) 1`

$$
\begin{aligned}
&\texttt{(rec f (fn x (if x 1 f(x-1)))) 1}\\
&\rightarrow \{\texttt{(rec f (fn x (if x 1 f(x-1))))}/\texttt{f}\}\{\texttt{1}/\texttt{x}\}\texttt{(if x 1 f(x-1))}\\
&\ \ = \texttt{if 1 1 (rec f (fn x (if x 1 f(x-1))))(1-1)}\\
&\rightarrow \texttt{(rec f (fn x (if x 1 f(x-1))))(1-1)}\\
&\rightarrow \texttt{(rec f (fn x (if x 1 f(x-1)))) 0}\\
&\rightarrow \{\texttt{(rec f (fn x (if x 1 f(x-1))))}/\texttt{f}\}\{\texttt{0}/\texttt{x}\}\texttt{(if x 1 f(x-1))}\\
&\ \ = \texttt{if 0 1 (rec f (fn x (if x 1 f(x-1))))(0-1)}\\
&\rightarrow \texttt{1}
\end{aligned}
$$

문제 3

이 언어의 문법은 아래와 같이 정의된다.

$$
\begin{array}{rcl}
E & \rightarrow & n\\
& | & x\\
& | & \texttt{fn}\, x\, E\\
& | & \texttt{rec}\, f\, x\, E\\
& | & E\, E\\
& | & \texttt{catch}\, x\, E\\
& | & \texttt{throw}\, E\, E
\end{array}
$$

무엇 의미구조는 다음과 같다.

$$
\begin{aligned}
[\![n]\!]\ \sigma\,\kappa &= \kappa\, n \\
[\![x]\!]\ \sigma\,\kappa &= \kappa\,(\sigma\, x) \\
[\![\texttt{fn}\, x\, E]\!]\ \sigma\,\kappa &= \kappa\,(\lambda v.\, \lambda k'.\ [\![E]\!]\ (\sigma\{x \mapsto v\})\, k') \\
[\![\texttt{rec}\, f\, x\, E]\!]\ \sigma\,\kappa &= \kappa\,(\mathrm{lfp}(\lambda v.\, \lambda v'.\, \lambda k'.\, [\![E]\!]\ (\sigma\{f \mapsto v\}\{x \mapsto v'\})\, k')) \\
[\![E_1\ E_2]\!]\ \sigma\,\kappa &= [\![E_1]\!]\ \sigma\,(\lambda f.\ [\![E_2]\!]\ \sigma\,(\lambda v.\, f\, v\, \kappa)) \\
[\![\texttt{catch}\, x\, E]\!]\ \sigma\,\kappa &= [\![E]\!]\ (\sigma\{x \mapsto \kappa\})\,\kappa \\
[\![\texttt{throw}\, E_1\ E_2]\!]\ \sigma\,\kappa &= [\![E_1]\!]\ \sigma\,(\lambda \kappa'.\ [\![E_2]\!]\ \sigma\,\kappa')
\end{aligned}
$$

문제 4

값은 다음과 같이 정의된다.

$$
\begin{array}{rcl}
v & \to & n \\
 & | & \texttt{fn}\, x\, E \\
 & | & \texttt{rec}\, f\, x\, E
\end{array}
$$

실행문맥(evaluation context) K 는 다음과 같이 정의된다.

$$
\begin{array}{rcl}
K & \to & [\cdot] \\
 & | & K\ E \\
 & | & v\, K \\
 & | & \texttt{throw}\, K\ E \\
 & | & \texttt{throw}\, v\, K
\end{array}
$$

실행의미는 다음과 같다.

$$
\begin{aligned}
K[(\texttt{fn}\, x\, E)\, v] &\to K[\{v/x\}E] \\
K[(\texttt{rec}\, f\, x\, E)\, v] &\to K[\{(\texttt{rec}\, f\, x\, E)/f\}\{v/x\}E] \\
K[\texttt{catch}\, x\, E] &\to K[\{(\texttt{fn}\, y\, K[y])/x\}E] \qquad y \notin \mathrm{Var}(K) \\
K[\texttt{throw}\, (\texttt{fn}\, x\, E)\, v] &\to \{v/x\}E
\end{aligned}
$$

문제 5

(a) 의미구조

$$
\begin{aligned}
[\![n]\!]\,\sigma &= n \\
[\![x]\!]\,\sigma &= \sigma(x) \\
[\![\texttt{-}E]\!]\,\sigma &= -([\![E]\!]\,\sigma) \\
[\![\texttt{let}\, x\, E_1\, E_2]\!]\,\sigma &= [\![E_2]\!]\ \sigma\{x \mapsto [\![E_1]\!]\ \sigma\}
\end{aligned}
$$

(b) CPS 변환

각 E에 대해 $\overline{E}$는 다음과 같이 정의된다.

$$
\begin{aligned}
\overline{n} &= \lambda k.\, k\, n \\
\overline{x} &= \lambda k.\, k\, x \\
\overline{\texttt{-}E_1} &= \lambda k.\, \overline{E_1}\, (\lambda x.\, k\, (-x)) \\
\overline{\texttt{let}\, x\, E_1\, E_2} &= \lambda k.\, \overline{E_1}\, (\lambda x.\, \overline{E_2}\, k)
\end{aligned}
$$

단, $k \notin \mathrm{Var}(E)$.

(c) CPS 변환의 올바름

정리 3.

$$\forall E, \sigma, \kappa.\ \kappa\ ([\![E]\!]\,\sigma) = [\![\overline{E}]\!]\ \sigma\ \kappa$$

Proof. E의 구조에 대한 귀납법으로 증명한다.

- $E = n$.

$$
\begin{aligned}
[\![\overline{n}]\!]\,\sigma\,\kappa &= [\![(\lambda k.\, k\, n)]\!]\,\sigma\,\kappa \\
&= \kappa\, n \\
&= \kappa\, ([\![n]\!]\,\sigma)
\end{aligned}
$$

- $E = x$.

$$
\begin{aligned}
[\![\overline{x}]\!]\,\sigma\,\kappa &= [\![(\lambda k.\, k\, x)]\!]\,\sigma\,\kappa \\
&= \kappa\, (\sigma\, x) \\
&= \kappa\, ([\![x]\!]\,\sigma)
\end{aligned}
$$

- $E = \texttt{-}E_1$.

$$\begin{aligned}
[\![\overline{\texttt{-}E_1}]\!]\,\sigma\,\kappa &= [\![(\lambda k.\, \overline{E_1}\,(\lambda x.\, k\,(-x)))]\!]\,\sigma\,\kappa \\
&= [\![\overline{E_1}]\!]\,\sigma\{k \mapsto \kappa\}\,(\lambda v.\, \kappa\,(-v)) \\
&= [\![\overline{E_1}]\!]\,\sigma\,(\lambda v.\, \kappa\,(-v)) \quad (\because k \notin \mathrm{Var}(E_1)) \\
&= \kappa\,(-\,[\![E_1]\!]\,\sigma) \quad (\because \text{귀납가정}) \\
&= \kappa\,([\![\texttt{-}E_1]\!]\,\sigma)
\end{aligned}$$

- $E = \texttt{let}\, x\, E_1\, E_2$.

$$\begin{aligned}
[\![\overline{\texttt{let}\, x\, E_1\, E_2}]\!]\,\sigma\,\kappa &= [\![(\lambda k.\, \overline{E_1}\,(\lambda x.\, \overline{E_2}\, k))]\!]\,\sigma\,\kappa \\
&= [\![\overline{E_1}]\!]\,\sigma\{k \mapsto \kappa\}\,(\lambda v.\ [\![\overline{E_2}]\!]\,\sigma\{k \mapsto \kappa\}\{x \mapsto v\}\,\kappa) \\
&= [\![\overline{E_1}]\!]\,\sigma\,(\lambda v.\ [\![\overline{E_2}]\!]\,\sigma\{x \mapsto v\}\,\kappa) \quad (\because k \notin \mathrm{Var}(E_1), \mathrm{Var}(E_2)) \\
&= [\![\overline{E_1}]\!]\,\sigma\,(\lambda v.\ [\![\overline{E_2}]\!]\,\sigma\{x \mapsto v\}\,\kappa) \\
&= (\lambda v.\ [\![\overline{E_2}]\!]\,\sigma\{x \mapsto v\}\,\kappa)\,([\![E_1]\!]\,\sigma) \quad (\because \text{귀납가정}) \\
&= [\![\overline{E_2}]\!]\,\sigma\{x \mapsto [\![E_1]\!]\,\sigma\}\,\kappa \\
&= \kappa\,([\![E_2]\!]\,\sigma\{x \mapsto [\![E_1]\!]\,\sigma\}) \quad (\because \text{귀납가정}) \\
&= \kappa\,([\![\texttt{let}\, x\, E_1\, E_2]\!]\,\sigma)
\end{aligned}$$

따라서 모든 E에 대해 성립한다. $\square$